DEMONSTRATIVE GEOMETRY

BY

GOLAK NATH CHATTERJEE, B.A.

FORMERLY SCHOLAR OF CHRIST'S COLLEGE, CAMBRIDGE
PROFESSOR OF MATHEMATICS, GOVERNMENT
COLLEGE, LAHORE

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PREFACE

THE Panjab University has adopted, for the Matriculation, the following scheme of examination in Geometry, which has been recommended by the University of Cambridge —

The paper in Geometry shall contain questions on Practical and on Theoretical Geometry. Every candidate shall be expected to answer questions in both branches of the subject

The questions on Practical Geometry shall be set on the constructions contained in the annexed Schedule A, together with easy extensions of them In cases where the validity of a construction is not obvious, the reasoning by which it is justified may be required Every candidate shall provide himself with a ruler graduated in inches and tenths of an inch, and in centimetres and millimetres, a set square, a protractor, compasses, and a hard pencil All figures should be drawn accurately. Questions may be set in which the use of the set square or of the protractor is forbidden

The questions on Theoretical Geometry shall consist of theorems contained in the annexed Schedule B, together with questions upon these theorems, easy deductions from them, and arithmetical illustrations Any proof of a Proposition shall be accepted which appears to the Examiners to form part of a systematic treatment of the subject; the order in which the theorems are stated in Schedule B is not imposed as the sequence of their treatment.

In the proof of theorems and deductions from them, the use of hypothetical constructions shall be permitted Proofs which are only applicable to commensurable magnitudes shall be accepted

SCHEDULE A

Bisection of angles and of straight lines

Construction of perpendiculars to straight lines.

Construction of an angle equal to a given angle

Construction of parallels to a given straight line

Simple cases of the construction from sufficient data of triangles and quadrilaterals

Division of straight lines into a given number of equal parts or into parts in any given proportions

Construction of a triangle equal in area to a given polygon

Construction of tangents to a circle and of common tangents to two circles

Simple cases of the construction of circles from sufficient data

Construction of a fourth proportional to three given straight lines and a mean proportional to two given straight lines

Construction of regular figures of 3, 4, 6, or 8 sides in or about a given circle

Construction of a square equal in area to a given polygon.

SCHEDULE B

Angles at a Point

If a straight line stands on another straight line, the sum of the two angles so formed is equal to two right angles, and the converse

If two straight lines intersect, the vertically opposite angles are equal.

Parallel Straight Lines

When a straight line cuts two other straight lines, if

(i.) a pair of alternate angles are equal,

or (ii.) a pair of corresponding angles are equal,

or (iii.) a pair of interior angles on the same side of the cutting line are together equal to two right angles,

then the two straight lines are parallel, and the converse

Straight lines which are parallel to the same straight line are parallel to one another

Triangles and Rectilinear Figures

The sum of the angles of a triangle is equal to two right angles.

If the sides of a convex polygon are produced in order, the sum of the angles so formed is equal to four right angles

If two triangles have two sides of the one equal to two sides of the other, each to each, and also the angles contained by those sides equal, the triangles are congruent

If two triangles have two angles of the one equal to two angles of the other, each to each, and also one side of the one equal to the corresponding side of the other, the triangles are congruent

If two sides of a triangle are equal, the angles opposite to these sides are equal; and the converse.

If two triangles have the three sides of the one equal to the three sides of the other, each to each, the triangles are congruent

If two right angled triangles have their hypotenuses equal, and one side of the one equal to one side of the other, the triangles are congruent

If two sides of a triangle are unequal, the greater side has the greater angle opposite to it; and the converse

Of all the straight lines that can be drawn to a given straight line from a given point outside it, the perpendicular is the shortest

The opposite sides and angles of a parallelogram are equal, each diagonal bisects the parallelogram, and the diagonals bisect one another

If there are three or more parallel straight lines, and the intercepts made by them on any straight line that cuts them are equal, then the corresponding intercepts on any other straight line that cuts them are also equal

Areas

Parallelograms on the same or equal bases and of the same altitude are equal in area

Triangles on the same or equal bases and of the same altitude are equal in area

Equal triangles on the same or equal bases are of the same altitude

Illustrations and explanations of the geometrical theorems corresponding to the following algebraical identities —

$$k(a+b+c+\quad)=ka+kb+kc+\quad,$$
$$(a+b)^2=a^2+2ab+b^2,$$
$$(a-b)^2=a^2-2ab+b^2,$$
$$a^2-b^2=(a+b)(a-b)$$

The square on a side of a triangle is greater than, equal to, or less than the sum of the squares on the other two sides, according as the angle contained by those sides is obtuse, right, or acute. The difference in the cases of inequality is twice the rectangle contained by one of the two sides and the projection on it of the other.

Loci

The locus of a point which is equidistant from two fixed points is the perpendicular bisector of the straight line joining the two fixed points

The locus of a point which is equidistant from two intersecting straight lines consists of the pair of straight lines which bisect the angles between the two given lines

The Circle

A straight line, drawn from the centre of a circle to bisect a chord which is not a diameter, is at right angles to the chord, conversely, the perpendicular to a chord from the centre bisects the chord

There is one circle, and one only, which passes through three given points not in a straight line

In equal circles (or, in the same circle) (i) if two arcs subtend equal angles at the centres, they are equal, (ii) conversely, if two arcs are equal, they subtend equal angles at the centres

In equal circles (or, in the same circle) (i) if two chords are equal, they cut off equal arcs, (ii) conversely, if two arcs are equal, the chords of the arcs are equal

Equal chords of a circle are equidistant from the centre, and the converse

The tangent at any point of a circle and the radius through the point are perpendicular to one another

If two circles touch, the point of contact lies on the straight line through the centres

The angle which an arc of a circle subtends at the centre is double that which it subtends at any point on the remaining part of the circumference

Angles in the same segment of a circle are equal, and, if the line joining two points subtends equal angles at two other points on the same side of it, the four points lie on a circle

The angle in a semicircle is a right angle, the angle in a segment

greater than a semicircle is less than a right angle; and the angle in a segment less than a semicircle is greater than a right angle.

The opposite angles of any quadrilateral inscribed in a circle are supplementary, and the converse

If a straight line touch a circle, and from the point of contact a chord be drawn, the angles which this chord makes with the tangent are equal to the angles in the alternate segments.

If two chords of a circle intersect either inside or outside the circle, the rectangle contained by the parts of the one is equal to the rectangle contained by the parts of the other.

Proportion Similar Triangles

If a straight line is drawn parallel to one side of a triangle, the other two sides are divided proportionally; and the converse

If two triangles are equiangular their corresponding sides are proportional; and the converse

If two triangles have one angle of the one equal to one angle of the other and the sides about these equal angles proportional, the triangles are similar

The internal bisector of an angle of a triangle divides the opposite side internally in the ratio of the sides containing the angle, and likewise the external bisector externally

The ratio of the areas of similar triangles is equal to the ratio of the squares on corresponding sides

This little book contains demonstrations of all the Theorems and Constructions enunciated in the above scheme, together with numerical illustrations and easy deductions

Besides the propositions given in the scheme, there are many others which the student will require in his mathematical studies, the simplest and most important of these have been printed in italics in the exercises, or the miscellaneous propositions in Book III.

This subject is intended to be studied after the student has gone through a preparatory course in Experimental Geometry, and it is taken for granted that he is already

familiar with the terms employed and has considerable practice in Geometrical Drawing In this connection it is well to remember that in the new regulations for the Matriculation examination Drawing and Mensuration are regarded as essential parts of Geometrical work.

In all examinations in Geometry the use of symbols and abbreviations is allowed, not only with the object of saving time in writing, but especially for the purpose of presenting the various parts of an argument in a concise and clear form.

In this book we shall use the following contractions.—

$\angle$	for *angle*		for *since*, or *because*
$\angle$s	,, *angles*	$=$	,, *is equal to*
line	,, *straight line.*	$>$	,, *is greater than*
st	,, *straight*	$<$	,, *is less than*
pt	,, *point*	alt.	,, *alternate*
sq	,, *square*	corresp	,, *corresponding*
$\bigcirc$	,, *circle*	hyp	,, *hypothesis*
Δ	,, *triangle*	const.	,, *construction*
$\perp$	,, *perpendicular*, or *is perpendicular to*	adj	,, *adjacent*
rt $\angle$	,, *right angle*	rect.	,, *rectangle*
$\parallel$	,, *is parallel to*	def	,, *definition*
$\parallel^{m}$	,, *parallelogram*	ax	,, *axiom*
	,, *therefore*	perp	,, *perpendicular*

In the answers at the end of the book I have ventured to give brief hints for the solution of the more difficult exercises, by which means I hope many students will be induced to solve these for themselves who would otherwise give them up after making very feeble efforts

GOLAK NATH CHATTERJEE

LAHORE, *July* 1906

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BOOK I

THEOREMS

BOOK II

BOOK III

MISCELLANEOUS PROPOSITIONS

INSTRUMENTS

A STRAIGHT-EDGE graduated in inches and tenths, and in centimetres and millimetres

Pencil compasses

Set-squares of 45° and 60°.

Protractor

Scissors

The above instruments should be in the hands of each student, and the teacher should keep in the school models of the various solid figures and their sections mentioned in the introductory chapter

Special and complete sets of mathematical instruments for use with this book can be obtained from the publishers, Messrs Rai Sahib M Gulab Singh & Sons, at exceptionally low prices to students and all educational bodies

BOOK I

INTRODUCTION

Demonstrative Geometry is the science which treats of the shape, size, and position of figures by pure reasoning on definitions, axioms, and established geometrical facts

The **definitions**, which state the exact meaning of geometrical terms, are given in an alphabetical list at the end of the book, which should be consulted by the student whenever he comes across a new term

The **axioms** are self-evident truths, such as—

Things which are equal to the same thing are equal to one another.

If equals be added to equals the sums are equal.

Doubles of the same thing, or of equal things, are equal.

The whole is greater than a part

A geometrical subject proposed for discussion is called a **proposition**

There are two kinds of propositions.—

(i) **Theorems**—in which geometrical facts are stated, and (ii) **Problems**—in which geometrical constructions are required to be made.

A theorem consists of two parts:—

(i) The **hypothesis**, or that which is assumed as true;

and (ii) the **conclusion**, or that which is asserted to follow from the hypothesis

Two theorems are said to be **converse**, each of the other, when the hypothesis of each is the conclusion of the other.

A **corollary** is a proposition the truth of which follows immediately from a proved proposition

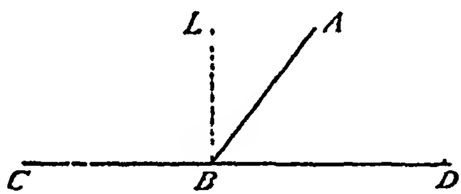
ANGLES AT A POINT

Proposition 1 —Theorem

If a straight line stands on another straight line, the sum of the two angles so formed is equal to two right angles.

Let the line AB stand on the line CD, so as to make with CD the $\angle$s ABC, ABD on the same side of CD, then

$$\angle DBA + \angle ABC = 2 \text{ rt. } \angle\text{s}$$



Let BL be drawn $\perp CD$ Then

$$\angle ABC = \angle LBC + \angle ABL,$$

and $$\angle DBA = \angle DBL - \angle ABL;$$

$$\therefore \angle ABC + \angle DBA = \angle LBC + \angle DBL$$
$$= 2 \text{ rt } \angle\text{s}. \qquad [Const$$

Cor 1 —*If two straight lines cut one another, the four angles so formed are together equal to four right angles*

Cor. 2 —*If any number of straight lines meet at a point, the sum of all the angles between successive lines is equal to four right angles*

EXERCISES

1. If a straight line stands on another straight line, the angles so formed are supplementary

2. *The bisectors of adjacent supplementary angles are at right angles to one another*

3. What angle is equal to its supplement?

4. Find the number of degrees in an angle which is one-third of its supplement.

5. Draw AB 3" long, and through its middle point O draw OC 1.5" long, and making the angle BOC of 43°; measure the angle AOC

6. In the figure of the last exercise draw OP, OQ, the bisectors of the angles BOC, AOC respectively; measure the angle POQ.

7. Two lines cut one another; if one of the angles so formed is a right angle, all the others are right angles also.

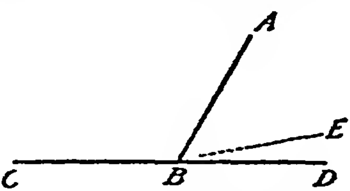
PROPOSITION 2 —THEOREM

If the sum of two adjacent angles is equal to two right angles, the external arms of the angles are in the same straight line

Let ABC, ABD be two adjacent angles such that

$$\angle DBA + \angle ABC = 2 \text{ rt. } \angle s,$$

then CB, BD are in the same straight line



If CB and BD are not in the same straight line, produce CB to E

Then ∵ CBE is a straight line and AB stands on it

$$\angle EBA + \angle ABC = 2 \text{ rt. } \angle s, \qquad \text{[Prop. 1}$$

and $$\angle DBA + \angle ABC = 2 \text{ rt. } \angle s; \qquad \text{[Hyp.}$$

$$\therefore \angle EBA + \angle ABC = \angle DBA + \angle ABC;$$
$$\therefore \angle EBA = \angle DBA,$$

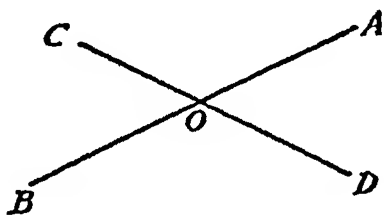
and this is impossible unless BE coincide with BD. Hence *BD and CB must be in the same straight line.*

Proposition 3—Theorem

If two straight lines intersect, the vertically opposite angles are equal.

Let the lines AB and CD cut one another at the point O; then

$$\angle DOA = \angle COB \text{ and } \angle AOC = \angle BOD$$



AO stands on COD,

$\therefore \angle DOA + \angle AOC = 2$ rt. $\angle$s. [*Prop* 1

And CO stands on AOB,

$\therefore \angle AOC + \angle COB = 2$ rt. $\angle$s. [*Prop* 1

Hence $\angle DOA + \angle AOC = \angle AOC + \angle COB$,

$\therefore \angle DOA = \angle COB$

Similarly, $\angle AOC = \angle BOD$

EXERCISES

1 Through a point O four straight lines, OA, OB, OC, OD, are drawn in order, such that

$$\angle BOC = \angle DOA, \text{ and } \angle AOB = \angle COD;$$

prove that AOC, BOD are straight lines

2 *The bisectors of vertically opposite angles are in the same straight line*

3 Draw two lines, intersecting one another at an angle of 45°, draw the bisectors of the four angles so formed

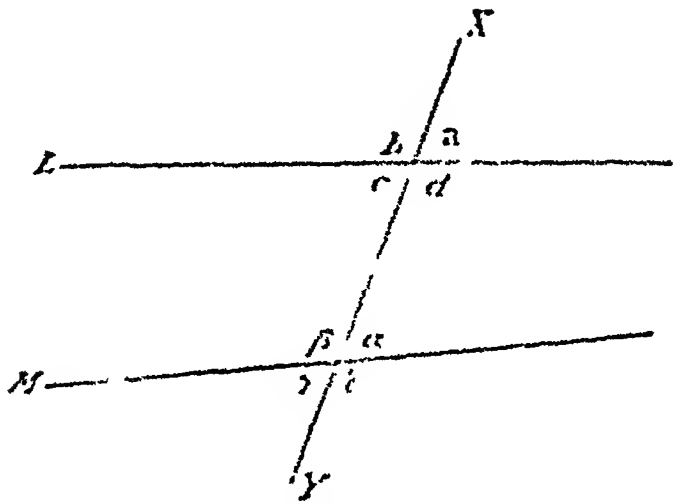
4 Draw any triangle BAC, and bisect the angle A, also bisect the external angle formed by producing BA, these two bisectors are called the internal and external bisectors of the angle A

Prove that *the internal and external bisectors of any angle of a triangle are at right angles*

5 Through any point O draw a line AOB, on opposite sides of AB make the angles BOC, BOD of 60° and 120° respectively Show that the non coincident arms of these angles are in the same straight line.

PARALLEL STRAIGHT LINES

When a straight line XY intersects two other straight lines L, M, it makes with them eight angles, which are named as follows,—



The angles a, b, γ, δ outside L and M are called **exterior** angles.

The angles d, c, β, α inside L and M are called **interior** angles.

A pair of interior angles on opposite sides of XY, such as (α, c) and (β, d), are called **alternate** angles.

A pair of angles on the same side of XY, one interior and the other exterior, are called **corresponding** angles.

In the figure there are four pairs of corresponding angles, viz (a, α), (b, β), (c, γ), and (d, δ)

In the treatment of parallel straight lines we shall make use of—

Playfair's axiom — *Two intersecting straight lines cannot both be parallel to the same straight line*

The student will find little difficulty in admitting this statement if he takes into consideration the fact that parallel straight lines have the same direction, while intersecting lines have different directions

PROPOSITION 4 -- THEOREM

When a straight line cuts two other straight lines, if

(i) *a pair of alternate angles are equal,*

or (ii) *a pair of corresponding angles are equal,*

or (iii) *a pair of interior angles on the same side of the cutting line are together equal to two right angles,*

then the two straight lines are parallel

Let the line LM cut the lines AB, CD in P and Q; then

(i) If the alternate angles α, β are equal, $AB \parallel CD$

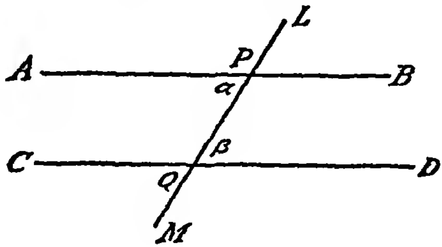


FIG. 1

Let Fig 2 be an exact reproduction of Fig 1.

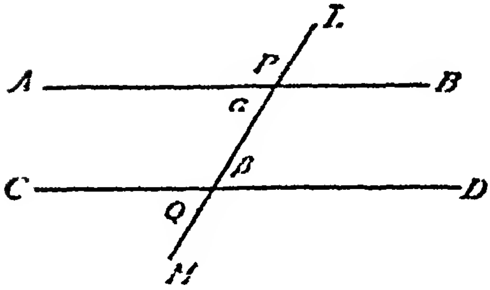


Fig. 2.

Place the second figure on the first in such a manner that the $\angle$s β, α of the second fit on the angles α, β respectively of the first.

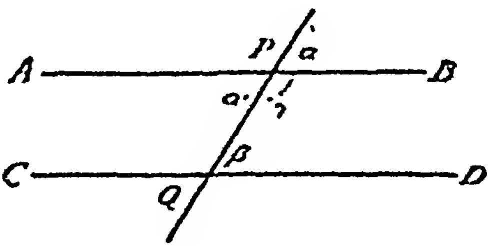
Then since the angles are equal their arms will coincide, so that DC will lie on AB and BA on CD.

In the doubled figure, if AB, CD meet, when produced, towards the right, they will also meet, when produced, towards the left; but this is impossible;

$\therefore$ AB, CD do not meet when produced.

Hence $AB \parallel CD$.

(ii) If the corresponding angles α, β are equal, $AB \parallel CD$.



The $\angle\alpha$ = vertically opposite $\angle\alpha'$; [*Prop* 3

$\therefore$ $\angle\beta$ = alternate $\angle\alpha'$;

$\therefore$ $AB \parallel CD$ [*Prop* 4, (i).

(iii) If the interior $\angle$s β, γ on the same side of LM are together equal to two right angles, $AB \parallel CD$.

QP meets AB,

$\angle\gamma + \angle\alpha' = 2$ rt $\angle$s, [Prop 1.

and $\angle\gamma + \angle\beta = 2$ rt $\angle$s, [Hyp.

$\angle\gamma + \angle\alpha' = \angle\gamma + \angle\beta$,

$\angle\alpha' = \angle\beta$

But these are alternate $\angle$s,

$AB \parallel CD$ [Prop 4, (i.)

PROPOSITION 5—THEOREM

If a straight line cut two parallel straight lines, it makes

(i) *alternate angles equal,*

(ii) *corresponding angles equal,*

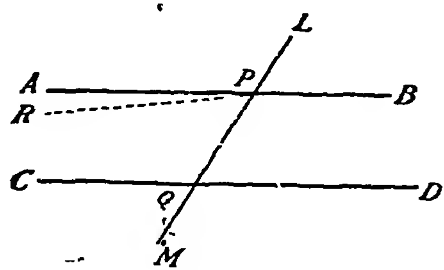
and (iii) *two interior angles on the same side of the cutting line supplementary*

Let the line LM cut the parallel lines AB, CD in P and Q, then

(i) $\angle APQ = \angle DQP$,

(ii) $\angle LPB = \angle PQD$,

and (iii.) $\angle BPQ + \angle DQP = 2$ rt $\angle$s



(i.) If $\angle APQ$ be not equal to $\angle DQP$,

let $\angle RPQ$ be equal to $\angle DQP$.

These are alternate angles,

$$\therefore RP \parallel CD, \quad \text{[Prop 4.}$$

two intersecting lines AP, RP are each $\parallel$ CD, which is impossible, [*Playfair's Axiom*

$$\angle APQ = \angle DQP$$

(ii) It has been proved that $\angle APQ = \angle DQP$, but $\angle APQ =$ the vertically opposite $\angle LPB$,

$$\therefore \angle LPB = \angle DQP$$

(iii) Since BP meets LM,

$$\angle BPQ + \angle LPB = 2 \text{ rt } \angle\text{s}. \quad \text{[Prop. 1.}$$

But $\angle LPB$ has been proved equal to the $\angle DQP$,

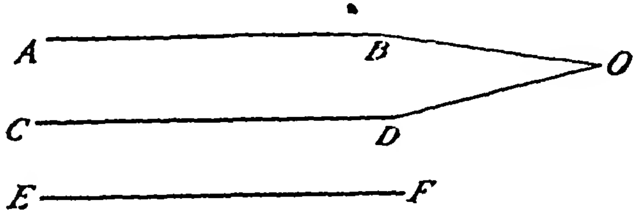
$$\angle BPQ + \angle DQP = 2 \text{ rt } \angle\text{s}$$

Proposition 6 —Theorem

Straight lines which are parallel to the same straight line are parallel to one another

Let the two lines AB, CD each be parallel to EF, then

$$AB \parallel CD.$$



For if AB and CD are not parallel, they will intersect if produced in some point O, and then the two intersecting lines OBA, ODC would be parallel to the same line EF, which is impossible; [*Playfair's Axiom.*

$$AB \parallel CD$$

EXERCISES

1 Deduce this Theorem from Propositions 4 and 5

2 Show that two straight lines at right angles to the same straight line are parallel

3 A straight line perpendicular to one of two parallel straight lines is also perpendicular to the other

4 If through any point parallels be drawn to two given intersecting lines, the acute angle between the parallels is equal to the acute angle between the given lines

5 When a straight line intersects two parallel straight lines, the four acute angles so formed are equal, and also the four obtuse angles

6 *Two angles which have their arms parallel are equal or supplementary*

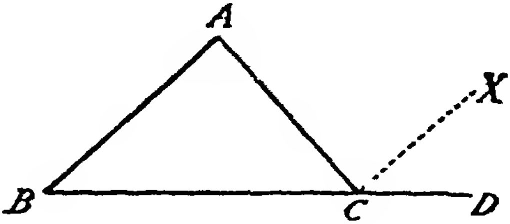
TRIANGLES AND RECTILINEAR FIGURES

PROPOSITION 7 —THEOREM

The sum of the angles of a triangle is equal to two right angles

Let ABC be any $\triangle$, then

$$\angle A + \angle B + \angle C = 2 \text{ rt } \angle\text{s}.$$



Produce BC to D, and draw $CX \parallel BA$;

$$\because CX \parallel BA,$$

$\angle A = \text{alt } \angle ACX,$ [*Prop* 5

and $\angle B = \text{corresp } \angle XCD,$ [*Prop* 5

$\therefore \angle A + \angle B = \angle ACD,$

$\angle A + \angle B + \angle C = \angle ACD + \angle BCA$

$= 2 \text{ rt } \angle\text{s}$ [*Prop* 1

COR —*If one side of a triangle is produced, the exterior angle is equal to the sum of the two interior opposite angles.*

$$(\angle ACD = \angle A + \angle B.)$$

Hence *the exterior angle is greater than either of the two interior opposite angles.*

EXERCISES

1 How many degrees are there in all the angles of a triangle?

2 A triangle cannot have more than one right angle, or more than one obtuse angle, but it may have three acute angles.

3 A right angled triangle has an angle of 60° Determine the other angles.

4 The exterior angle formed by producing one of the sides of a right angled triangle is of 126° Find all the angles.

5 The angles at the base of a triangle are equal, and the exterior angle at the vertex is of 120° Show that all the angles of the triangle are equal

6 Two angles of a right angled triangle are equal, how many degrees are there in each?

7 If one angle of a triangle is equal to the sum of the other two, the triangle is right angled

8 Through the vertex of any triangle draw a straight line parallel to the base, and by considering the angles at the vertex prove that the sum of the angles of the triangle is equal to two right angles.

9 One of the acute angles of a right angled triangle is double of the other, how many degrees does it contain?

10 Two triangles have two angles of the one equal to two angles of the other, each to each, show that their remaining angles are also equal.

11 The angles of a triangle are in the ratios 3 5 7. Express the angles in degrees

12 Show that Prop 1 is a particular case of Prop 7

PROPOSITION 8 —THEOREM

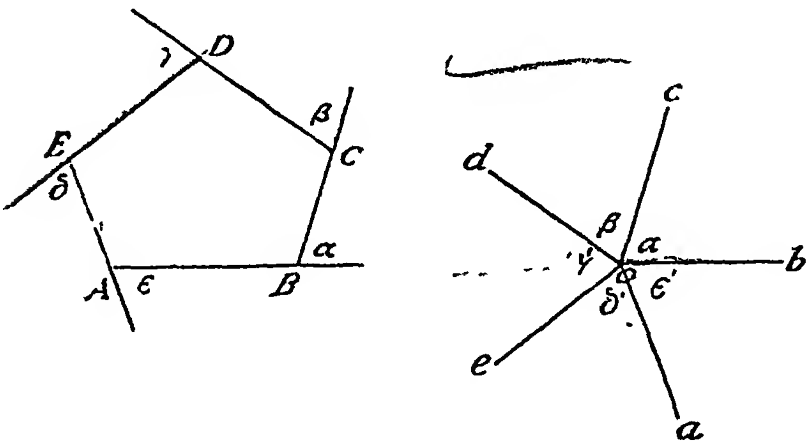
If the sides of a convex polygon are produced in order, the sum of the angles so formed is equal to four right angles

DEF —A **convex** polygon is one which has no angle greater than two right angles.

Let the sides of the convex polygon $ABCDE$ be produced in order, and form the exterior angles α, β, γ, δ, ϵ; then

$$\angle\alpha + \angle\beta + \angle\gamma + \angle\delta + \angle\epsilon = 4 \text{ rt. } \angle\text{s}.$$

Through any point O draw Ob, Oc, Od, Oe, Oa parallel to and in the same sense as AB, BC, CD, DE, and EA respectively



$\because$ Ob, Oc are $\parallel$ AB, BC, and drawn in the same sense,

$$\angle\alpha = \angle\alpha'.$$

Similarly,

$$\angle\beta = \angle\beta',$$

$$\angle\gamma = \angle\gamma',$$

$$\angle\delta = \angle\delta',$$

and

$$\angle\epsilon = \angle\epsilon',$$

$$\therefore \angle\alpha + \angle\beta + \angle\gamma + \angle\delta + \angle\epsilon = \angle\alpha' + \angle\beta' + \angle\gamma' + \angle\delta' + \angle\epsilon'.$$

But $\angle\alpha' + \angle\beta' + \angle\gamma' + \angle\delta' + \angle\epsilon' = 4$ rt. $\angle$s, [*Prop* 1, *Cor.* 2.

$$\therefore \angle\alpha + \angle\beta + \angle\gamma + \angle\delta + \angle\epsilon = 4 \text{ rt } \angle\text{s}$$

COR.—*The sum of all the interior angles of a convex polygon of n sides is $2n - 4$ right angles* For at each angular point there is an interior and an exterior angle whose sum is 2 rt. $\angle$s (*Prop.* 1); hence the sum of all the interior and exterior angles is $2n$ rt $\angle$s

But the sum of the exterior angles is 4 rt $\angle$s: therefore the sum of the interior angles is $2n - 4$ rt. $\angle$s.

EXERCISES

1 All the angles of a regular polygon are equal, if it has n sides find the number of degrees in each angle

2 How many degrees are there in an angle of a regular pentagon?

3 Determine the angles of a regular hexagon, octagon, decagon, and dodecagon

4 The exterior angle of a regular polygon is 60° How many sides has it?

5 The exterior angle of a regular polygon is 45° Determine the number of sides

6 A polygon has each of its angles of 144°, find the number of sides.

7 Each angle of a polygon is of 150°, how many sides has it?

8 Each of the angles of a polygon is $1\frac{1}{2}$ right angles Find the number of sides

9 In a convex quadrilateral the sum of the interior angles is equal to the sum of the exterior angles

10 In a convex hexagon the sum of the interior angles is equal to twice the sum of the exterior angles

11 Can you draw regular polygons having angles (1) 130°, (2) 140°, (3) 175° respectively? Determine the number of sides of the polygon if it can be drawn

PROPOSITION 9—THEOREM

If two triangles have two sides of the one equal to two sides of the other, each to each, and also the angles contained by those sides equal, the triangles are congruent

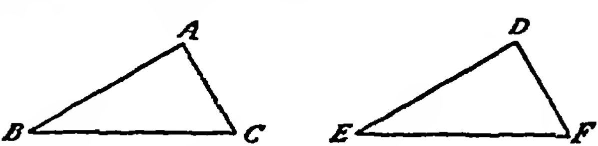
Let ABC, DEF be two $\triangle$s such that

$$AB = DE,\ AC = DF,$$

and

$$\angle BAC = \angle EDF,$$

then the two $\triangle$s are congruent



Place the $\triangle DEF$ on the $\triangle ABC$, so that DE lies

on its equal AB, the point D falling on A and the point E falling on B

Then $\angle D = \angle A$,

DF will lie in the direction AC;

and $\therefore DF = AC$,

F falls on C,

$\therefore \triangle DEF$ coincides with $\triangle ABC$,

hence the two $\triangle$s are congruent.

EXERCISES

1. With sides 5 cm and 8 cm, and included angle of 54°, draw two triangles and letter them as in the figure of this proposition, placing the letters inside the triangles Cut them out, and go through the process of superposition described above.

2 In the two quadrilaterals $ABCD$, $A'B'C'D'$, the sides AB, BC, CD are equal to the sides $A'B'$, $B'C'$, $C'D'$ respectively, and the angles at B, D are equal to the angles at B', D' respectively, prove by the method of superposition that the quadrilaterals are congruent.

3 Draw two lines 6 and 8 inches long, bisecting one another and containing an angle of 60°; prove that the straight lines joining the extremities of these lines form a parallelogram

4. If the straight lines AC, BD bisect one another at right angles, the figure $ABCD$ is a rhombus

5 From two lines AX, AY inclined at any angle, cut off $AB = AC$ and $AD = AE$; join BE and CD Prove that the triangles ABE, ACD are equal.

6 ABC is an isosceles triangle, D and E respectively are the middle points of the equal sides AB, AC; prove that the triangles ABE, ACD are congruent.

7. From the extremities of the base of a square, two lines are drawn to the middle point of the opposite side, show that they cut off equal triangles from the square.

8 On the sides of the triangle ABC equilateral triangles $A'BC$, $AB'C$, ABC' are described externally, prove that $AA' = BB' = CC'$.

9 *The diagonals of a rectangle are equal*

10 A rectangle is divided into four equal parts by its diagonals.

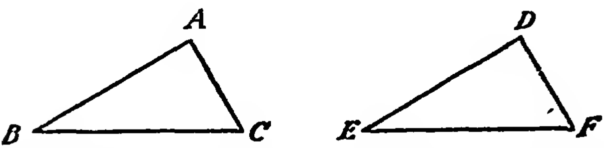
Proposition 10 —Theorem

If two triangles have two angles of the one equal to two angles of the other, each to each, and also one side of the one equal to the corresponding side of the other, the triangles are congruent

Let ABC, DEF be two $\triangle$s, such that

$$\angle B = \angle E \quad \angle C = \angle F, \text{ and } BC = EF,$$

then the $\triangle$s are congruent.



Place $\triangle DEF$ on $\triangle ABC$, so that EF lies on its equal BC, E falling on B and F falling on C, then

$\angle E = \angle B$, ED will take the direction BA,

and $\angle F = \angle C$, FD will take the direction CA

Hence ED and FD will meet where BA and CA meet,

D will fall on A,

$\triangle DEF$ will coincide with $\triangle ABC$,

the two $\triangle$s are congruent

EXERCISES

1 With base 3″, and angles at the base of 30° and 60°, draw two triangles, cut them out and go through the process of superposition described above

2. On a base BC 2 8′ long describe a triangle ABC having each angle at the base of 75°, draw BE, CF bisecting the angles at the base and meeting the opposite sides in E and F Prove that the triangles BEC, BFC are congruent

3 In the triangles ABC, DEF the bases BC and EF are equal,

and the angles A, B are equal to the angles D, E respectively; prove that the triangles are congruent.

4. The hypotenuse and acute angle of a right-angled triangle are equal respectively to the hypotenuse and acute angle of another right-angled triangle; prove that the triangles are equal in every respect.

5. Every point on the bisector of the angle contained by two lines is equidistant from the lines.

6. Draw a parallelogram with sides $3'$ and $4.5''$, and one angle of $75°$; draw the diagonals and prove that the opposite sides are equal.

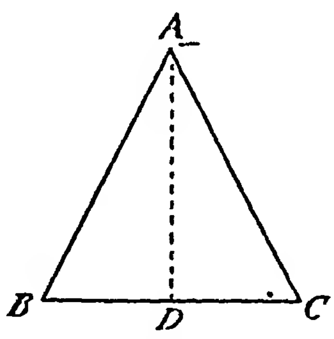
7. *Prove that the diagonals of a parallelogram bisect one another.*

8. A diagonal of a quadrilateral bisects the angles through which it passes; prove that the quadrilateral is a kite.

PROPOSITION 11.—THEOREM.

If two sides of a triangle are equal, the angles opposite to these sides are equal.

In the $\triangle ABC$ let $AC = AB$, then $\angle B = \angle C$.



Let AD be the bisector of the $\angle A$.

In the $\triangle$s ABD, ACD,

$$\begin{cases} AB = AC, & \text{[Hyp.} \\ AD = AD, & \\ \text{included } \angle BAD = \text{included } \angle CAD; & \text{[Const.} \end{cases}$$

$\therefore$ $\triangle$s ABD, ACD are congruent. [*Prop.* 9.

Hence $\angle B = \angle C$.

EXERCISES

1 Verify the proof by folding an isosceles triangle ABC about AD

2 *The bisector of the vertical angle of an isosceles triangle bisects the base at right angles*

3 *If the equal sides of an isosceles triangle be produced, the angles on the other side of the base will be equal to each other*

4 *An equilateral triangle is also equiangular*

5 The vertical angle of an isosceles triangle is of 30°, determine the angles at the base

6 The exterior angle at the vertex of an isosceles triangle is of 144°, calculate all the angles of the triangle

7 *The straight line bisecting the exterior angle at the vertex of an isosceles triangle is parallel to the base*

8 How many degrees are there in one of the angles of an equilateral triangle?

9 *Trisect a right angle*

10 If two isosceles triangles have their vertical angles equal, their base angles will also be equal

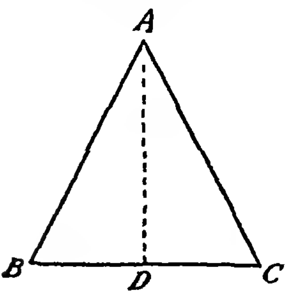
11 Determine the angles of a right isosceles triangle

12 AB is the diameter of a semicircle, and P is any point on the circumference, prove that the angle APB is a right angle

13 Join the extremities of any two diameters of a circle, and prove that the figure so formed is a rectangle

PROPOSITION 12 —THEOREM

If two angles of a triangle are equal, the sides opposite to these angles are equal



In the triangle ABC let $\angle B = \angle C$, then

$$AC = AB$$

Let AD be the bisector of the angle BAC

In the Δs ABD, ACD,

$$\begin{cases} \angle B = \angle C, & [Hyp \\ \angle BAD = \angle CAD, & [Const \\ AD = AD, & \end{cases}$$

$\therefore$ $\triangle$s ABD, ACD are congruent [*Prop.* 10

Hence $AB = AC$

COR.—*An equiangular triangle is also equilateral.*

EXERCISES

1 If OB, OC, the bisectors of the angles B, C of the triangle ABC, be equal to one another, the triangle is isosceles

2. If the bisector of the exterior vertical angle of a triangle be parallel to the base, the triangle is isosceles

3 Two sides of a triangle are produced, and the exterior angles so formed are equal, prove that the triangle is isosceles.

4. Through any point on the bisector of an angle a straight line is drawn parallel to one of the containing sides, prove that the triangle so formed is isosceles

5 The bisectors of the base angles of an isosceles triangle form with the base another isosceles triangle

6. The bisectors of the base angles of an equilateral triangle meet in O Through O parallels are drawn to the sides; prove that these parallels trisect the base

PROPOSITION 13—THEOREM

If two triangles have the three sides of the one equal to the three sides of the other, each to each, the triangles are congruent.

Let ABC, DEF be two $\triangle$s in which $AB = DE$, $BC = EF$, $CA = FD$; then

$\triangle$s ABC, DEF are congruent.

Of the sides of $\triangle ABC$, let BC be not less than either of the other two.

$\because$ $EF = BC$, $\triangle DEF$ can be placed in the position LBC, so that EF and BC coincide, E falling on B and F on C, and L is the position of D.

Join AL, meeting BC in O.

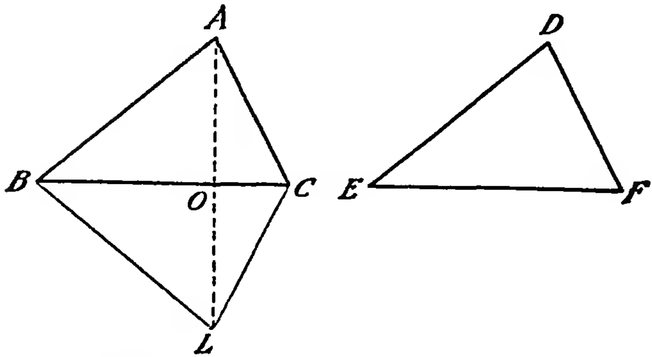
$BA = ED = BL$, $\angle BAO = \angle BLO$ [*Prop* 11.

Also

$CA = FD = CL$, $\angle CAO = \angle CLO$, [*Prop* 11.

$$\angle BAC = \angle BLC$$

Hence $\angle BAC = \angle EDF$



In $\triangle$s ABC, DEF,

$$\begin{cases} AB = DE, \\ AC = DF, \\ \text{and } \angle BAC = \angle EDF, \end{cases}$$

$\triangle$s ABC, DEF are congruent [*Prop* 9

EXERCISES

1 On the same base two equilateral triangles are described, prove that the line joining their vertices bisects the base

2 Show that either diagonal of a rhombus divides it into two congruent triangles, and that the diagonals bisect one another at right angles

3 On the base AB two unequal isosceles triangles, ACB, ADB, are constructed, prove that CD (produced if necessary) bisects AB at right angles

4 In the isosceles triangle ABC the equal angles are bisected by the lines BO, CO, prove that AO bisects the vertical angle

5. Equilateral triangles on equal bases are congruent

6. *The straight line, which joins the vertex of an isosceles triangle to the middle point of the base, is at right angles to the base and bisects the vertical angle*

PROPOSITION 14.—THEOREM

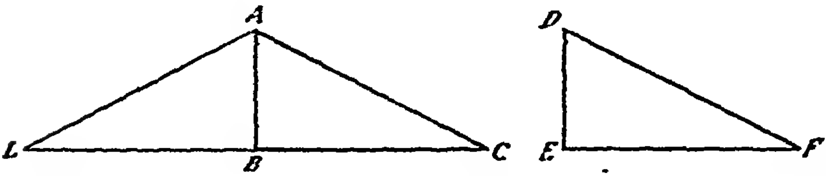
If two right-angled triangles have their hypotenuses equal, and one side of the one equal to one side of the other, the triangles are congruent

Let ABC, DEF be two $\triangle$s right-angled at B and E, and having

$$AC = DF \text{ and } AB = DE,$$

then $\triangle$s ABC, DEF are congruent.

$\because DE = AB$, the triangle DEF can be placed in the



position ABL so that DE coincides with its equal AB and F falls at the point L on the side of AB opposite to C.

$\because$ the adjacent angles ABC, ABL are rt $\angle$s.

$\therefore$ LBC is a st line. [*Prop* 2.

In $\triangle ALC$, $AC = DF = AL$:

$\therefore \angle ACB = \angle ALB$. [*Prop* 11.

Hence in $\triangle$s ABC, ABL,

$$\because \begin{cases} \angle ABC = \angle ABL, & [\textit{Def.} \\ \angle ACB = \angle ALB, & [\textit{Proved} \\ AB = AB; & \end{cases}$$

$\therefore$ $\triangle$s ABC, ABL are congruent, [*Prop.* 10.

$\therefore$ $\triangle$s ABC, DEF are congruent.

EXERCISES

1. *Two right angled triangles are equal when they have the hypotenuse and an acute angle equal*

2. *Two right-angled triangles are equal when they have a side and an acute angle equal*

3. The perpendicular, from the vertex of an isosceles triangle on the base, bisects the base.

4. The perpendicular, from the centre of a circle on any chord, bisects the chord

5. If the perpendiculars, from the base angles of a triangle on the opposite sides, are equal, the triangle is isosceles.

6. If the three perpendiculars, from the angular points of a triangle on the opposite sides, are equal, the triangle is equilateral

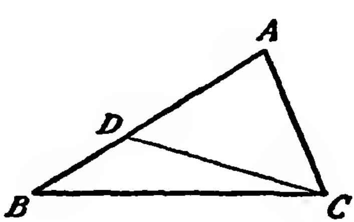
7. A point P lies within the angle AOB, and the perpendiculars from P on OA, OB are equal, prove that OP bisects the angle AOB.

PROPOSITION 15 —THEOREM

If two sides of a triangle are unequal, the greater side has the greater angle opposite to it

Let ABC be a $\triangle$ in which $AB > AC$, then

$$\angle C > \angle B$$



From AB cut off $AD = AC$ Join CD.

In $\triangle ACD$, $AC = AD$,

$\therefore \angle ACD = \angle ADC$ [Prop 11.

But $\because$ the side BD of the triangle BDC is produced to A,

$\therefore$ exterior $\angle ADC >$ interior opposite $\angle B$, [*Prop.* 7, *Cor.*

$$\therefore \angle ACD > \angle B;$$

$$\therefore \angle C > \angle B.$$

EXERCISES

1 In the figure of this proposition describe a circle with centre A and radius equal to the lesser side AC, cutting CB in R, join AR. Then noticing that

$$\angle ARC > \angle B, \qquad \text{[}Prop\ 7,\ Cor$$

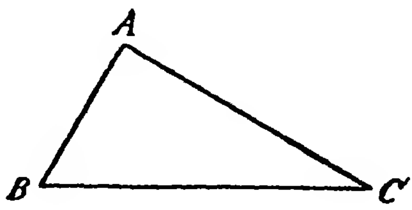
prove the proposition

2 Prove this proposition by producing the lesser side and cutting it off equal to the greater

PROPOSITION 16 —THEOREM

If two angles of a triangle are unequal, the greater angle has the greater side opposite to it

In $\triangle ABC$ let $\angle B > \angle C$, then $AC > AB$.



With respect to magnitude either (i.) $AC > AB$,

or (ii.) $AC = AB$,

or (iii.) $AC < AB$.

If $AC<AB$, then

$$\angle B<\angle C, \qquad [Prop\ 15$$

which is contrary to hypothesis

Again, if $AC=AB$, then

$$\angle B=\angle C, \qquad [Prop\ 11$$

which also is contrary to hypothesis

$$AC>AB$$

EXERCISES

1 *Two sides of a triangle are together greater than the third*

In the figure of this proposition, the straight path from B to C must be shorter than the crooked path $BA+AC$

2 Prove that the circumference of a circle is greater than twice the diameter

3 From a given point to a given line three equal lines could not be drawn

4 *The difference between two sides of a triangle is less than the third*

5 If O be a point within the triangle ABC, then

$$OA+OB+OC>\tfrac{1}{2}(BC+CA+AB)$$

6 If O be a point within the triangle ABC, then

$$AB+AC>OB+OC,$$

and

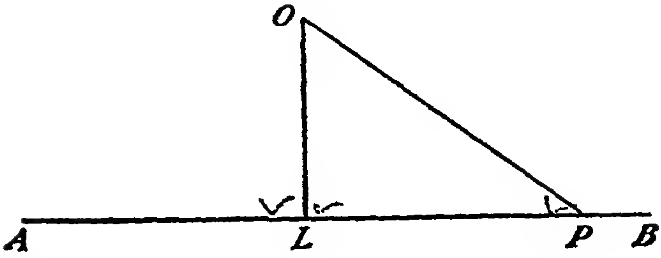
$$AB+BC+CA>OA+OB+OC,$$

Proposition 17 —Theorem

Of all the straight lines that can be drawn to a given straight line from a given point outside it, the perpendicular is the shortest

Let AB be the given line, and O a point outside it, OL is drawn perpendicular to AB, and OP is any other line through O meeting AB in P, then

$$OL<OP$$



In $\triangle OLP$,

$\because$ PL is produced to A,

$\therefore$ exterior $\angle OLA >$ interior opposite $\angle OPL$, [Prop 7, Cor.

but $\angle OLA = \angle OLP$, [Def

$\therefore$ $\angle OLP > \angle OPL$,

$OL < OP$ [Prop. 16

Note.—The distance between a point and a straight line is the shortest line that can be drawn from the point to the line Hence the perpendicular, from a given point on a given line, is called the **distance** of the point from the line

EXERCISES

1 In the figure of this proposition prove that—

(i) *Oblique lines which make equal angles with the perpendicular are equal*

(ii) *Of two oblique lines, that is longer which deviates more from the perpendicular*

2 *Two and only two oblique lines, of given length, can be drawn from a given point to a given straight line Give a construction for this and show when it fails*

3 *A circle cannot cut a straight line in more than two points*

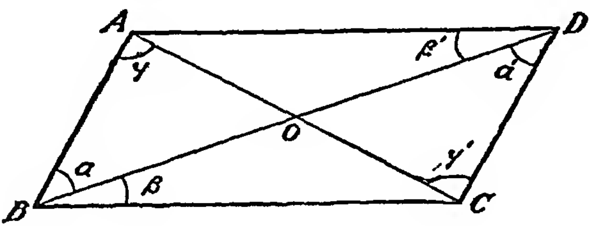
4 A straight line joining the vertex of an isosceles triangle to any point in the base is less than either of the equal sides, but greater if the point be on the base produced.

5 *Any two sides of a triangle are together greater than twice the line joining the vertex to the middle point of the base*

PROPOSITION 18—THEOREM

The opposite sides and angles of a parallelogram are equal, each diagonal bisects the parallelogram, and the diagonals bisect one another

Let $ABCD$ be a $\|^m$, then its opposite sides and angles are equal, the diagonals AC and BD each bisect it and bisect one another at their point of intersection O



(i) In Δs BAD, BCD,

$$\begin{cases} \angle\alpha = \angle\alpha', & [Alt\ \angle s. \\ \angle\beta = \angle\beta', & [Alt\ \angle s. \\ BD = BD, & \end{cases}$$

Δs are congruent. [*Prop* 10.

Hence $AB = CD$, $AD = BC$,

$\angle BAD = \angle BCD$,

and $\Delta ABD = \Delta BCD$.

Similarly, from the Δs ABC, ADC, it can be proved that

$$\angle ABC = \angle CDA,$$

and that AC bisects the figure

(ii) In Δs OAB, OCD,

$$\begin{cases} \angle\alpha = \angle\alpha', & [Alt\ \angle s. \\ \angle\gamma = \angle\gamma', & [Alt\ \angle s. \\ AB = CD, & [Proved. \end{cases}$$

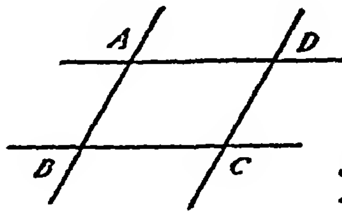
$\therefore$ Δs are congruent. [*Prop.* 10.

Hence $OA = OC$ and $OB = OD$.

EXERCISES

1. Two parallels AB, CD included between two other parallels AD, BC are equal.

2. *Two parallels are everywhere equally distant.*



3. *Prove that a quadrilateral is a parallelogram if*
   (i) *one pair of opposite sides are equal and parallel;*
   (ii) *pairs of opposite sides are equal;*
   (iii) *pairs of opposite angles are equal;*
   (iv) *the diagonals bisect each other.*

4. In the figure of this proposition any line through O, terminated by either pair of opposite sides, is bisected at O and divides the parallelogram into two equal quadrilaterals.

5. If one angle of a parallelogram is a right angle, all its angles are right angles.

6. If one pair of adjacent sides of a parallelogram are equal, all its sides are equal.

7. Prove that all the angles of a square are right angles and all its sides are equal.

8. The diagonals of a rectangle are equal.

9. The diagonals of a rhombus, and of a square, bisect one another at right angles.

10. If the diagonals of a parallelogram are equal, it is a rectangle.

11. If the diagonals of a parallelogram are equal and perpendicular, it is a square.

12. The bisectors of the four angles of a parallelogram enclose a rectangle.

PROPOSITION 19.—THEOREM

If there are three or more parallel straight lines, and the intercepts made by them on any straight line that cuts them

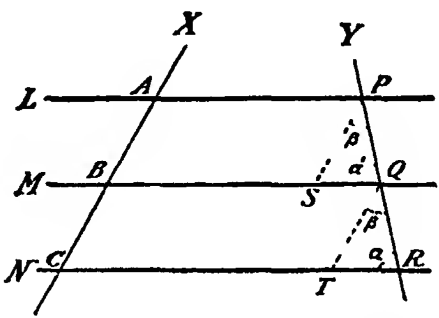
are equal, then the corresponding intercepts on any other straight line that cuts them are also equal

Let L, M, N be parallel lines, and let a line X cut them in A, B, C, making

$$AB = BC,$$

then, if any other line Y cuts them in P, Q, R,

$$PQ = QR$$



Let PS, QT be drawn parallel to X, meeting M, N in S, T respectively

Then AS is a $\|^m$,

$$PS = AB, \qquad \text{[Prop 18}$$

and BT is a $\|^m$,

$$QT = BC \qquad \text{[Prop 18}$$

But $AB = BC$, [*Hyp*

$$PS = QT$$

Now $M \parallel N$,

$$\angle\alpha = \text{corresp } \angle\alpha'$$

Again, PS, QT are each $\parallel X$,

$$PS \parallel QT, \qquad \text{[Prop 6.}$$

$$\angle\beta = \text{corresp } \angle\beta'$$

Hence in Δs PSQ, QTR,

$$\because \begin{cases} \angle\alpha' = \angle\alpha, \\ \angle\beta' = \angle\beta, \\ PS = QT; \end{cases} \qquad [\textit{Prop. } 10.$$

Δs are congruent,

$\therefore \quad PQ = QR.$

EXERCISES

1 In the triangle ABC the side AB is divided into any number of equal parts, and through the points of division parallels are drawn to the base BC, prove that these parallels will divide the side AC into the same number of equal parts.

2 *A straight line drawn through the middle point of the side of a triangle parallel to the base will bisect the other side.*

3 *The straight line joining the middle points of the sides of a triangle is parallel to the base and equal in length to half the base*

4 *The straight lines joining the middle points of the adjacent sides of a quadrilateral form a parallelogram*

5 *The straight lines joining the middle points of opposite sides of a quadrilateral bisect one another.*

MISCELLANEOUS QUESTIONS AND EXERCISES.—I.

1 What is Demonstrative Geometry?

2 What is a Theorem? What are the two essential parts of a theorem? Give examples

3 When are two theorems said to be converse, each of the other? Point out all such pairs of theorems in the first nineteen propositions.

4 Define a *point*, what are its positive and negative geometrical properties respectively?

5 Draw two *adjacent* angles of 30° and 135°

6 Define a *right angle* How would you show practically that the angular space, about any point in a plane, contains four right angles?

7 What is meant by the *complement* of an angle? Draw two adjacent complementary angles, one of them being of 30°

8 Why does not the size of an angle depend upon the lengths of its arms?

9 Define *supplementary* angles Given an angle, what is the simplest construction for finding its supplement?

10 Define *parallel straight lines*, and explain *Playfair's Axiom*

11 "*Parallel straight lines are such as lie in the same plane, and which remain at the same distance from each other, however far they are produced*"

From what proposition can this be deduced as a corollary?

12 Under what circumstances can two straight lines be produced ever so far without meeting, and yet not be parallel?

13 If two straight lines be not parallel, prove that all straight lines cutting them make alternate angles whose difference is constant

14. Define a *straight line*, and from your definition deduce that—

Any two sides of a triangle are together greater than the third

15 Can you construct a triangle with any three given lengths for sides?

16 Draw a line cutting two others, and point out the pairs of *corresponding angles* it makes with them

17 Define a *triangle*, how many different kinds of triangles are there? Draw one of each kind

18 In an acute-angled triangle any two of the acute angles are together greater than the third

19 Three sides of any quadrilateral are together greater than the fourth

20 The sum of the interior angles of a rectilineal figure is double of the sum of its exterior angles, find the number of sides

21 Show that if the straight lines bisecting the angles B, C of the triangle ABC meet in O, then OA will bisect the angle A

22 If AA', BB', CC be three diameters of a circle, prove that the triangles ABC, $A'B'C$ are congruent

23 From the sides BC, CA, AB of an equilateral triangle, lengths BP, CQ, AR, each equal to one-third of a side of the triangle, are cut off Show that the triangle PQR is equilateral, and has its sides perpendicular to the sides of the triangle ABC

24 Prove that any side of a triangle is less than half the perimeter.

25 If the bisector of an angle of a triangle bisect also the opposite side, the triangle is isosceles

26 Define *parallelogram*, *rectangle*, *rhombus*, and *square*.

27 Draw a parallelogram whose diagonals are 5 6 cm and 7 6 cm. respectively, and contain an angle of 45°

28 Draw a rhombus whose diagonals are 3 8" and 4.8" respectively.

29 Construct a square whose diagonal is 3 5"

30 Draw a rectangle whose diagonal is 5″ long and makes an angle of 30° with a side, show that the shorter side is half the length of the diagonal.

31 From the base angles of an isosceles triangle perpendiculars are drawn to the sides, show that the angles made by them with the base are each equal to half the vertical angle

32 In the quadrilateral *ABCD*, the lines drawn from *A* to the other angular points are all equal Show that the angle *BAD* is double of the sum of the angles *CBD* and *CDB*

33 If two parallelograms have a common angle, show that all their angles are equal each to each

34 *ABC* is an equilateral triangle; *AC* is produced to *D*, so that *CD* is equal to *CA*, and *BD* is joined Show that *BD* is perpendicular to *AB*.

35 Take *AB* 3″ long, and without producing *AB* draw a straight line at right angles to it through the point *B*

36 In the isosceles triangle *ABC* the bisectors of the equal angles, *B* and *C*, meet the opposite sides in *D* and *E* respectively, prove that *BE*, *CD*, *DE* are equal

37. In the triangle *ABC* the bisectors of the angles at *B* and *C* meet in *I*; *DE* is drawn through *I* parallel to *BC* and meeting *AB*, *AC* in *D* and *E* respectively Show that *DE* is equal to the sum of *BD* and *CE*

38 Draw a parallelogram whose diagonals are 10 cm and 12 cm, and one side is 7 cm

39 Explain what is meant by the *distance* of a point from a straight line. A point being taken on one of the arms of an angle of 30°, show that its distance from the other arm is equal to half its distance from the vertex of the angle

40 If the diagonal of a parallelogram bisect the angles through which it passes, the parallelogram is a rhombus

41 The bisectors of the base angles of an isosceles triangle contain an angle equal to an exterior angle at the base of the triangle

42 A straight line drawn at right angles to *BC*, the base of an isosceles triangle *ABC*, cuts the equal sides *AB*, *AC* in *D* and *E* respectively; prove that *ADE* is an isosceles triangle

43 Each of the interior angles of a rectilineal figure is of 160°, find the number of sides

44. Each of the exterior angles of a polygon is of 18°; find the number of sides

45 The join of the middle points of two opposite sides of a quadri-

D

lateral is at right angles to each of these sides, show that the other two sides are equal

46 Prove that the hypotenuse of a right-angled triangle is greater than either of its sides

47 The sum of the sides of a quadrilateral is greater than the sum of its diagonals

48 In an obtuse angled triangle the greatest side is opposite to the obtuse angle.

49 On a base BC 2 7″ long describe the triangle ABC, having the angles at B and C of 36° and 84° respectively, let the bisector of the angle A meet BC in D, prove that AD is greater than BD

50 Prove that the bisectors of the base angles of any triangle can never be at right angles to each other

51 If the middle points of the three sides of a triangle be joined, the triangle is divided into four triangles, which are equal in every respect.

52 The straight lines drawn through the angular points of the sides of a triangle, parallel to the opposite sides, form another triangle whose sides are twice as long as the sides of the given triangle

53 The middle points of the sides of a triangle are at distances of 2 6″, 2 8″, and 1 7″ from one another, draw the triangle

54 Take a line AB 5 7″ long, and trisect it in the points D and E; on DE describe an equilateral triangle DEF, and join FA, FB Prove that the angle AFB is of 120°

55 What regular polygons have their exterior angles right angles and obtuse angles respectively?

56 The interior angle of a regular polygon is seventeen times the exterior angle, find the number of sides

57 Define an *acute angled triangle*, and prove that the perpendicular from any of the angles of an acute angled triangle on the opposite side falls within the triangle

58 The perpendicular from either of the acute angles of an obtuse-angled triangle on the opposite side falls outside the triangle

59 What is a *scalene triangle*? Show that all its angles are unequal

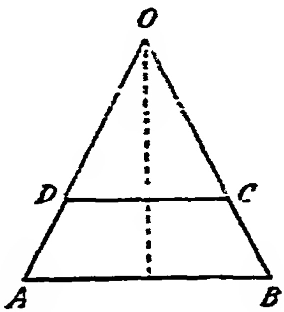
60 If two angles of a triangle be unequal, the smaller of the two must be acute

Define a *trapezium* A straight line DC is drawn parallel to the base of an isosceles triangle OAB, forming the trapezium $ABCD$. Prove the following properties, 61–64 —

61 The non parallel sides are equal

62. The diagonals are equal.

63 The line joining the middle points of the parallel sides is perpendicular to both of them.



64. The line joining the middle points of the non-parallel sides bisects both diagonals, and is perpendicular to the line of Ex. 63.

65 One of the non-parallel sides BC of the trapezium $ABCD$ is bisected in O, and through O the straight line FOE is drawn parallel to AD, meeting AB, DC in F and E respectively; prove that

$$AF=DE=\tfrac{1}{2}(AB-CD).$$

66 Draw a line 7.8 cm. long, and divide it into six equal parts.

67. Take a line 5.5" long: divide it by construction into five equal parts, and check by measurement.

68 The sides AB, CD of the parallelogram $ABCD$ are bisected in E and F; show that ED, BF trisect the diagonal AC.

69. If any point on the diagonal of a square is joined to its angular points, the square is divided into two pairs of equal triangles.

70 The straight line joining the middle points of the non-parallel sides of a trapezium is equal to half the sum of the parallel sides.

71. The straight line joining the middle points of the diagonals of a trapezium is equal to half the difference of the parallel sides.

72. Define a *kite*.

A kite $ABCD$ is constructed by describing two unequal isosceles triangles ACB, ACD on opposite sides of the same base AC; prove that BD bisects the whole figure, and bisects AC at right angles.

73 The internal bisectors of the angles of a parallelogram form a rectangle, whose diagonals are parallel to the sides of the original parallelogram.

74. In the sides AB, BC, CD, DA of a square $ABCD$, points

P, Q, R, S are taken so that $AP=BQ=CR=SD$, prove that $PQRS$ is a square

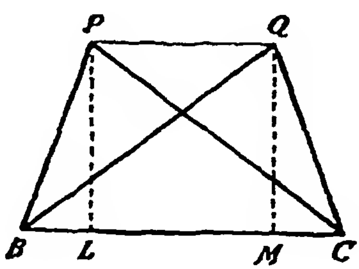
75 The sides BC, CA, AB of the equilateral triangle ABC are produced to P, Q, R respectively, so that $CP=CB$, $AQ=AC$, and $BR=BA$, prove that PQR is an equilateral triangle.

76 Prove that if the middle points of the sides of a rectangle, taken in order, be joined, the figure so formed is a rhombus, and if the middle points of the sides of a rhombus be similarly joined the resulting figure is a rectangle.

AREAS

DEF.—*The* altitude *of a parallelogram or triangle, with reference to a given side selected as* base, *is the perpendicular distance between the base and the opposite side or vertex*

From this definition it follows at once that if two parallelograms, or two triangles, stand on the same base and have the same altitude, they are between the same parallels.



For, taking the case of two Δs PBC, QBC which stand on the same side of a base BC, and have equal altitudes PL and QM, we see that

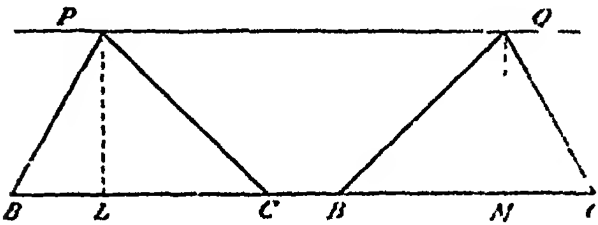
$\because$ PL, QM are equal and $\parallel$, [*Hyp*

$\therefore$ $PQML$ is a $\parallel^{m}$. [*Prop* 18, *Ex* 3

Hence the Δs PBC, QBC are between the same parallels.

In the same way it may be shown that if two Δs stand

on equal bases, which are in the same straight line, and have equal altitudes, they are between the same parallels



The student will find no difficulty in making the corresponding figures for the case of two parallelograms

Again, it should be noticed that if two triangles, or two parallelograms, have equal bases and equal altitudes, they can always be placed as in the first figure, where they stand on the same base and have the same altitude

In the propositions which follow, the student will come across figures which although equal in area are not equal in all respects

Such figures are equivalent but not congruent

PROPOSITION 20—THEOREM

Parallelograms on the same base and of the same altitude are equal in area

Let $ABCD$, $ABPQ$ be $\|^{ms}$ on the same base AB, and having the same altitude, then the $\|^{ms}$ are equal in area.

The altitudes of the $\|^{ms}$ are equal,

$\therefore$ they are between the same $\|^s$ [*Prop* 18, *Ex* 3

Hence $CPDQ$ is a st. line

Then $\because BC \| AD$,

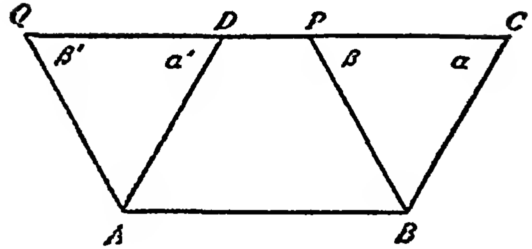
$\therefore \angle a =$ corresp $\angle a'$; [*Prop.* 5.

and $\because BP \parallel AQ$,

$\therefore \angle\beta = \text{corresp } \angle\beta'$, [Prop. 5.

also $\because BD$ is a $\parallel^m$,

$BC = AD$ [Prop 18.



Hence in Δs BCP, ADQ,

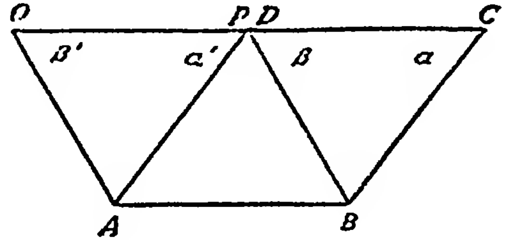
$$\because \begin{cases} \angle\alpha = \angle\alpha', \\ \angle\beta = \angle\beta', \\ BC = AD, \end{cases}$$

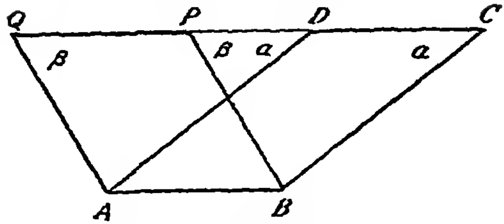
$\therefore$ Δs BCP, ADQ are congruent and equal in area. [Prop 10

Taking each in turn from the whole figure $ABCQ$, the remainders, viz. the $\parallel^{ms}$ $ABPQ$, $ABCD$, are equal in area.

EXERCISES

1. *Show that the above proof applies equally well to the following two figures —*





2 *Parallelograms on equal bases and of the same altitude are equal in area*

3 *Equal parallelograms on equal bases have the same altitude*

4 *Equal parallelograms having the same altitude stand on equal bases*

5 *The area of a parallelogram is equal to that of a rectangle having the same base and height*

6 With sides 5 cm and 7 cm, and included angle of 30°, draw a parallelogram construct a rectangle equal to it in area.

7 The straight lines joining the middle points of the sides of a triangle form with those sides four parallelograms which are equal in area.

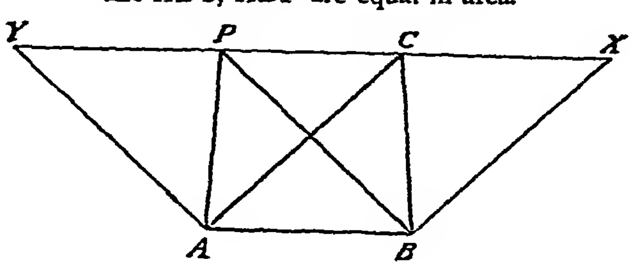
8 Divide a given parallelogram into five parallelograms of equal area.

Proposition 21 —Theorem

Triangles on the same base and of the same altitude are equal in area

Let ABC, ABP be two $\triangle$s on the same base AB, and of the same altitude then

$\triangle$s ABC, ABP are equal in area.



$\because$ $\triangle$s ABC, ABP have the same altitude,
they are between the same $\|^s$ [*Prop* 18, *Ex* 3

Draw BX, $AY \parallel AC$, BP respectively, meeting PC produced in X, Y, then $XCPY$ is $\parallel$ to AB, and AX, BY are $\|^{ms}$ on the same base AB and between the same $\|^s$,

$\therefore \|^{m} AX = \|^{m} BY.$ [*Prop* 20

But $\|^{m} AX = 2\triangle ABC,$ [*Prop* 18

and $\|^{m} BY = 2\triangle ABP,$

$\therefore \triangle ABC = \triangle ABP$

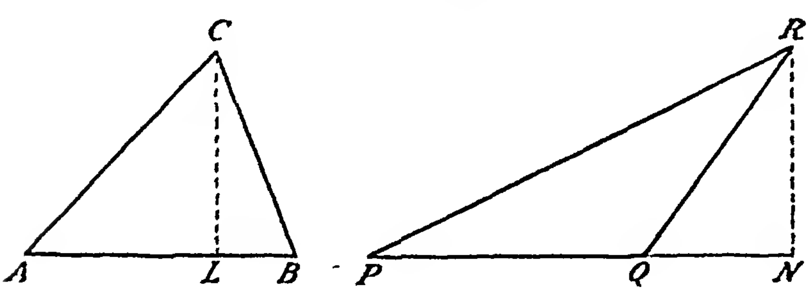
COR 1 —*If a parallelogram and a triangle be on the same base and have the same altitude, the area of the triangle is equal to half that of the parallelogram*

COR 2 —*The area of a triangle is equal to half that of a rectangle having the same base and height*

PROPOSITION 22 —THEOREM

Equal triangles on equal bases are of the same altitude

Let ABC, PQR be two $\triangle$s of equal area on equal bases AB, PQ, then their altitudes CL, RN are equal



$\triangle ABC = \frac{1}{2}AB \cdot CL,$ [*Prop* 21, *Cor* 2.

$\triangle PQR = \frac{1}{2}PQ \cdot RN$ [*Prop*. 21, *Cor* 2.

But $\triangle ABC = \triangle PQR$, [Hyp.

$\therefore \frac{1}{2}AB\,CL = \frac{1}{2}PQ\,RN$,

and $\because AB = PQ$, [Hyp.

$\therefore CL = RN$

EXERCISES

1. *Triangles on the same base and between the same parallels are equal*

2 *Triangles on equal bases and between the same parallels are equal.*

3 *Equal triangles on the same base, and on the same side of it, are between the same parallels*

4. *Triangles of equal area on equal bases in the same straight line, and on the same side of it, are between the same parallels*

5 The straight line drawn from the vertex of a triangle to the middle point of the base bisects the triangle

6 Show how to divide a triangle into three equivalent triangles by straight lines drawn from the vertex

7 Of two unequal triangles between the same parallels, that which has the longer base has the greater area

8 A triangle and a parallelogram have equal altitudes, but the base of the parallelogram is half of the base of the triangle, show that the areas are equal

9 *Two triangles have two sides of the one equal to two sides of the other, each to each, and the contained angles are supplementary, prove that their areas are equal*

10 *Two triangles of equal area are on the opposite sides of the same base, prove that the straight line joining their vertices is bisected by the base*

11. If one diagonal of a quadrilateral bisects another it also bisects the quadrilateral

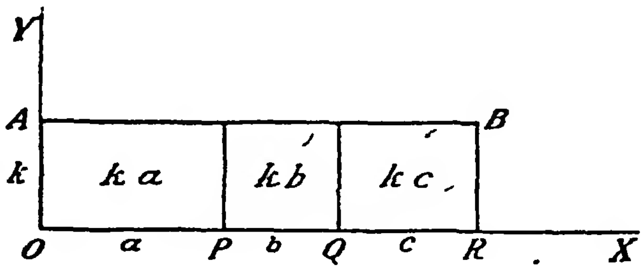
12. The diagonals of a parallelogram divide it into four equal triangles.

Proposition 23 ✓

Explanation of the geometrical theorem corresponding to the algebraical identity

$$k(a+b+c) = ka + kb + kc.$$

Take two st lines OX, OY at rt. ∠s Cut off OP, PQ, QR lengths of a, b, c units, and cut off OA a length of k units



Complete the rectangles as shown in the figure Then the whole rectangle $ORBA$ contains

$$k(a+b+c) \text{ units of area},$$

and its three parts contain

$$ka,\ kb,\ kc \text{ units of area respectively.}$$

Hence the corresponding geometrical theorem is :—

If there be two straight lines, one of which is divided into any number of parts, the rectangle contained by the two straight lines is equal to the sum of the rectangles contained by the undivided line and the several parts of the divided line

EXERCISES

1 Construct a geometrical figure to illustrate the algebraical identity

$$(a+b)(c+d)=ac+bc+ad+bd$$

2 Draw a figure corresponding to the identity

$$a(a+b)=a^2+ab.$$

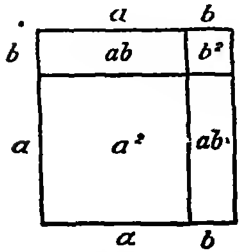
Proposition 24

Explanation of the geometrical theorem corresponding to the algebraical identity

$$(a+b)^2=a^2+2ab+b^2.$$

Take a st line $a+b$ units in length, and on it describe a square

On the part a units describe another square, and produce its sides to meet the first square as in the diagram.



The whole figure is a square containing

$(a+b)^2$ units of area,

also, of its parts the smaller squares contain

a^2 and b^2 units of area respectively,

and each rectangle contains

ab units of area

Hence the corresponding geometrical theorem is —

If a straight line be divided into two parts, the square on the whole line is equal to the sum of the squares on the two parts together with twice the rectangle contained by the parts.

EXERCISES

1. Draw a figure corresponding to the identity

$$(a+b)^2=a(a+b)+b(a+b)$$

2. Give a geometrical illustration of the identity

$$(a+b+c)^2=a^2+b^2+c^2+2bc+2ca+2ab$$

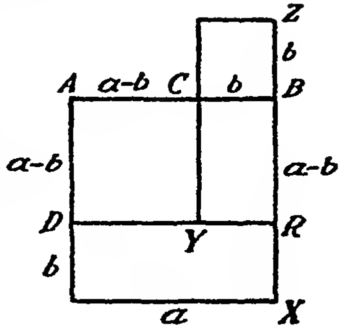
PROPOSITION 25

Explanation of the geometrical theorem corresponding to the algebraical identity

$$(a-b)^2=a^2-2ab+b^2$$

Take AB of length a units, and from it cut off BC, of length b units

On AB, AC, BC describe the squares AX, AY, CZ as in the figure, and produce DY to R.



The rectangles DX, YZ each contain ab units of area

The whole figure contains $a^2 + b^2$ units of area, and is made up of the two rectangles DX, YZ and the square AY, hence

$$a^2 + b^2 = 2ab + (a - b)^2,$$

i.e. $(a - b)^2 = a^2 - 2ab + b^2$

Hence the corresponding geometrical theorem is :—

If a straight line be divided into any two parts, the sum of the squares on the whole line and one of the parts is equal to twice the rectangle contained by the whole line and that part together with the square on the other part

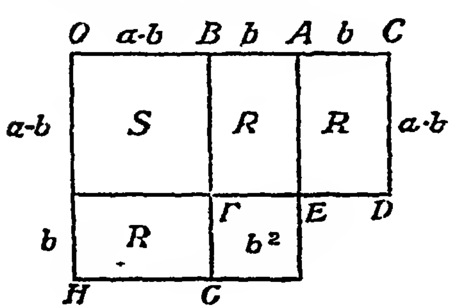
EXERCISE

Construct the above figure, taking $a = 5''$ and $b = 2''$

PROPOSITION 26

Explanation of the geometrical theorem corresponding to the algebraical identity

$$a^2 - b^2 = (a + b)(a - b).$$



Take OA of length a units, and on either side of A cut off AB, AC each b units of length On OB, OA describe squares and complete the diagram

The rectangles R are all equal.

Also rectangle $OD = OC \cdot CD$

$$= (a+b)(a-b),$$

$$\therefore S + R + R = (a+b)(a-b)$$

Again, the figure $OAEFGHO = a^2 - b^2$,

$$\therefore S + R + R = a^2 - b^2.$$

Hence $a^2 - b^2 = (a+b)(a-b)$

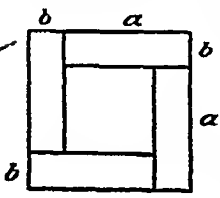
The corresponding geometrical theorem therefore is :—

The rectangle contained by the sum and difference of two straight lines is equal to the difference of the squares on the lines

EXERCISES

1 Construct the above diagram, taking a and b five and three inches respectively

2 In the following diagram the several lines are perpendicular and parallel, and the lengths of some of them are given, show that it illustrates the algebraical identity



$$(a+b)^2 - (a-b)^2 = 4ab$$

3 Draw a figure illustrating the identity

$$(x+a)(x+b) = x^2 + ax + bx + ab$$

4 If the unit of length be a centimetre, and x be any given length, draw a diagram corresponding to the identity

$$(x+4)(x+5) = x^2 + 9x + 20$$

5 In the figure of Ex. 2 produce the sides of the inner square both ways, and use the figure so formed to prove the identity

$$(a+b)^2 + (a-b)^2 = 2a^2 + 2b^2$$

6 *If A, B, C, D be four points in a straight line, prove that*

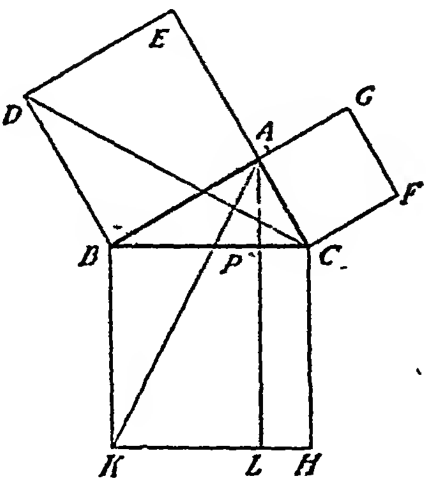
$$AB \cdot CD + BC \cdot AD = AC \cdot BD$$

PROPOSITION 27.—THEOREM

The square described on the hypotenuse of a right-angled triangle is equal to the sum of the squares described on the other two sides

Let ABC be a right-angled triangle, BAC being the right angle; then

$$BC^2 = CA^2 + AB^2$$



On BC, CA, AB describe the squares BH, CG, AD

Draw $AL \parallel CH$ or BK and meeting HK in L.

Join AK and CD

$\because$ $\angle$s BAC and BAE are rt $\angle$s,

CAE is a st line [*Prop.* 2.

Similarly, BAG is a st line

Now rt $\angle ABD$ = rt $\angle CBK$

To each of these add $\angle ABC$, then

$\angle CBD = \angle ABK$.

Then in $\triangle$s CBD, ABK,

$\because \begin{cases} BD = AB, & \text{[Const.} \\ BC = BK, & \text{[Const} \\ \text{and } \angle CBD = \angle ABK, & \text{[Proved.} \end{cases}$

$\therefore$ $\triangle CBD = \triangle ABK$ [*Prop* 9.

But sq. $AD = 2\triangle CBD$, [*Prop.* 21, *Cor.* 1.

and $\|^m\ BL = 2\triangle ABK$,

hence sq $AD = \|^m\ BL$

Similarly, by joining AH, BF, it can be proved that

$$\text{sq } CG = \|^m\ CL$$

But $\|^m\ BL + \|^m\ CL = \text{sq } BH$,

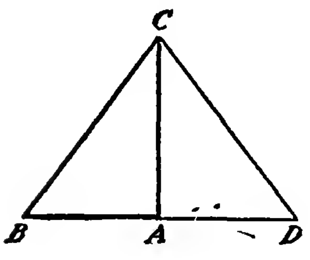
hence sq $BH = \text{sq } CG + \text{sq } AD$;

i e $BC^2 = CA^2 + AB^2$

COR 1 —*In a right-angled triangle, if a perpendicular be drawn from the right angle to the hypotenuse, the square on a side of the triangle is equal to the rectangle contained by the hypotenuse and its segment adjacent to that side*

$$(AB^2 = BC \cdot BP, \text{ and } AC^2 = CB \cdot CP)$$

COR. 2 —*If the square on one side of a triangle be equal to the sum of the squares on the other two sides, the angle contained by these sides is a right angle*



In $\triangle ABC$, let

$$BC^2 = AC^2 + AB^2$$

Draw $AD \perp AC$, and equal in length to AB, join CD. Then in $\triangle CAD$,

$\angle A$ is a rt $\angle$,

$CD^2 = AC^2 + AD^2$ [*Prop* 27.

But $AD = AB$, [*Const*

$\therefore\ CD^2 = AC^2 + AB^2$

$= BC^2$ [*Hyp*

$\therefore\ CD = BC$

Again, in Δs CAD, CAB,

$$\begin{cases} CD = BC, & \text{[Proved.} \\ AD = AB, & \text{[Const} \\ AC = AC, & \end{cases}$$

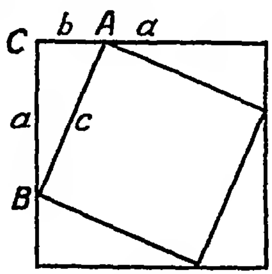
Δs are congruent, [Prop 13

and $\angle BAC = \angle DAC$,

which is a right angle

Note.—Proposition 27 is known as the **Theorem of Pythagoras**

The following proof of this important Theorem given by Hindu mathematicians will interest the readers



The area of the larger square

$$= a^2 + b^2 + 2ab$$

$$= c^2 + 4\Delta ABC$$

But area of ABC is $\frac{1}{2}\,ab$,

$$\therefore\ a^2 + b^2 + 2ab = c^2 + 2ab,$$

$$a^2 + b^2 = c^2$$

This proof can be explained to the class by folding a square piece of paper as shown in the figure

EXERCISES

1 Find a square equal to the sum of two given squares

2 Construct a square equal to the difference of two given squares.

3 Take two equal lengths OA, OP at right angles to one another; from OA produced cut off $OB = PA$, $OC = PB$, $OD = PC$, $OE = PD$, and so on, show that if OA represent unity, then PA, PB, PC . represent the square roots of the numbers 2, 3, 4, 5 respectively

4. Taking one centimetre as the unit of length, draw a line whose length is $\sqrt{(7)}$ centimetres

5 The square described on the diagonal of a square is double the original square

6. Find a square which is equal to the sum of three given squares

7 *The square described on a straight line is four times the square described on half the line*

8 Three times the square on a side of an equilateral triangle is equal to four times the square on its altitude

In the figure of Proposition 27 prove the following theorems 9–15 —

9 AD and AF are in the same straight line

10 BE is parallel to CG

11 AK is perpendicular to CD, and AH is perpendicular to BF

12 Angles ABC, KBD are supplementary

13 Angles ACB, FCH are supplementary

14 Triangles DBK, FCH are each of them equal to the triangle ABC

15 The triangle EAG equals the triangle ABC

16 *In the straight line AB, or AB produced, a point L is taken, and LP is drawn at right angles to AB, show that the difference of the squares on PA and PB is equal to the difference of the squares on AL and BL*

17 *If a perpendicular is drawn from the vertex of a triangle on the base, then the difference of the squares on the sides of the triangle is equal to the difference of the squares on the segments of the base*

18 The sides of a triangle are represented by the numbers 11, 60, and 61, show that it has a right angle

19 The sides of a triangle are represented by the numbers 2, $\sqrt{3}$ and 1, show that it is a right angled triangle having an acute angle of 60°

20 If the difference of the squares on two sides of a triangle be equal to the square on the third side, the triangle is right angled.

DEF 1 —*The foot of the perpendicular through a given point to a given straight line is called the projection of the given point on the given straight line*

In the figure of Prop 27 the point P is the projection of A on BC When the given point is on the given straight line it coincides with its projection on it. Thus in the figure of Prop 27 the projections of the points C and B on AB are A and B respectively

DEF 2.—*The projection of a given straight line on another is the straight line joining the projections of its extremities*

In the figure of Prop 27, BP and CP are the projections of AB and AC on BC

Notation—The sides of the triangle ABC, which are opposite to the angles A, B, C respectively, are denoted by the corresponding small letters a, b, c

PROPOSITION 28—THEOREM

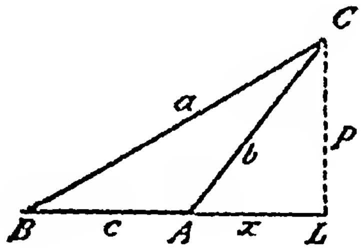
In an obtuse angled triangle, the square on the side opposite to the obtuse angle is equal to the sum of the squares on the other two sides together with twice the rectangle contained by either of these sides and the projection upon it of the other

Let ABC be a triangle having the obtuse angle BAC.

Draw $CL(=p)$ perpendicular to BA produced, then $AL(=x)$ is the projection of AC on BA.

It is required to prove that

$$a^2 = b^2 + c^2 + 2cx$$



$\because$ L is a right angle,

$$a^2 = (c + x)^2 + p^2, \qquad \text{[Prop. 27.}$$

and $$b^2 = x^2 + p^2, \qquad \text{[Prop. 27}$$

$$\therefore a^2 - b^2 = (c + x)^2 - x^2$$

$$= c^2 + 2cx,$$

$$\therefore a^2 = b^2 + c^2 + 2cx.$$

EXERCISES

1. Draw a triangle with sides 7 cm, 5 cm, and 4 cm; calculate the length of the perpendicular drawn from the opposite angle on the side 4 cm Hence find the area of the triangle.

2 Draw a triangle with sides 1", 1 7", and 2 1", calculate the length of the perpendicular from the opposite angle on the shortest side Hence find the area of the triangle.

3 The side of an equilateral triangle is unity, find the length of the perpendicular from the vertex on the base, also find the lengths of the segments of the base made by the perpendicular

4 The line AB is inclined at an angle of 60° to the line AX, prove that the projection of AB on AX is equal to half of AB

5 In the triangle ABC the angle A is of 120°, prove that

$$a^2 = b^2 + c^2 + bc$$

6 Draw the triangle ABC, having given $b = 4''$, $c = 3''$, and $\angle A = 120°$, find a.

7 Construct the triangle ABC, in which $b = 4\ 6''$, $c = 3\ 8''$, and $A = 150°$, calculate the lengths of the perpendiculars from the acute angles on the opposite sides

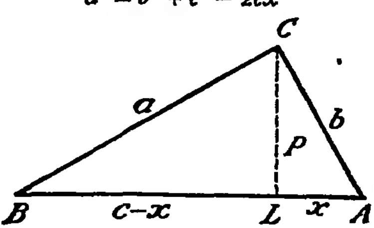
Proposition 29—Theorem

In any triangle the square on the side opposite to an acute angle is less than the sum of the squares on the sides containing the acute angle by twice the rectangle contained by either of those sides and the projection upon it of the other side

In the $\triangle ABC$ let BAC be an acute angle Draw $CL(=p)$ perpendicular to BA, then $AL(=x)$ is the projection of CA on AB

It is required to prove that

$$a^2 = b^2 + c^2 - 2cx$$



$\angle L$ is a rt $\angle$,

$$a^2 = (c-x)^2 + p^2, \qquad [Prop\ 27$$

and $$b^2 = x^2 + p^2,$$ [*Prop.* 27.

$$\therefore a^2 - b^2 = (c-x)^2 - x^2$$
$$= c^2 - 2cx,$$
$$\therefore a^2 = b^2 - c^2 - 2cx.$$

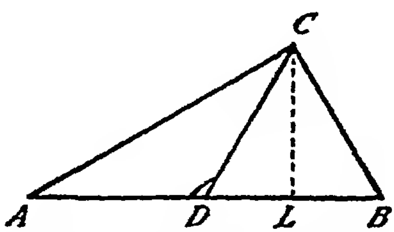
COR.—*In any triangle the sum of the squares on two sides is equal to twice the square on half the third side, and twice the square on the median that bisects the third side.*

DEF.—The straight lines drawn from the angular points of a triangle to the middle points of the opposite sides are called the **medians** of the triangle.

In the $\triangle ABC$ let CD be the median which bisects AB in D, then

$$AC^2 - BC^2 = 2AD^2 + 2CD^2$$

Draw $CL \perp AB$



Then $$AC^2 = AD^2 + CD^2 - 2AD.DL;$$ [*Prop.* 28.

and $$BC^2 = BD^2 + CD^2 - 2BD.DL$$ [*Prop.* 29

Since $AD = BD$, we have by addition

$$AC^2 + BC^2 = 2AD^2 + 2CD^2.$$

EXERCISES

1 With sides 7 cm, 8 cm., and 9 cm draw a triangle; calculate the lengths of the segments of the longest side made by the perpendicular drawn to it from the opposite angle.

2 In the triangle ABC the angle A is of 60°, prove that

$$a^2 = b^2 + c^2 - bc$$

3 Draw the triangle ABC, having given $b=7$ cm, $c=5$ cm, and $A=60°$, calculate the length of a

4 In the figure of Prop 29 prove that

$$AL=\frac{b^2+c^2-a^2}{2c} \text{ and } BL=\frac{c^2+a^2-b^2}{2c}$$

5 In the figure of Prop 29 calculate the length of the perpendicular CL.

We have

$$\begin{aligned}
p^2&=b^2-x^2\\
&=b^2-\left(\frac{b^2+c^2-a^2}{2c}\right)^2\\
&=\frac{4b^2c^2-(b^2+c^2-a^2)^2}{4c^2}\\
&=\frac{(2bc+b^2+c^2-a^2)(2bc-b^2-c^2+a^2)}{4c^2}\\
&=\frac{\{(b+c)^2-a^2\}\{a^2-(b-c)^2\}}{4c^2}\\
&=\frac{(b+c+a)(b+c-a)(a+b-c)(a-b+c)}{4c^2};
\end{aligned}$$

$$\therefore \quad p=\frac{1}{2c}\sqrt{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}.$$

Now let s denote the semi-perimeter of ΔABC, then

$$a+b+c=2s,$$
$$b+c-a=a+b+c-2a=2s-2a=2(s-a),$$
$$c+a-b=a+b+c-2b=2s-2b=2(s-b),$$
and
$$a+b-c=a+b+c-2c=2s-2c=2(s-c)$$

Substituting these in the expression for p we get

$$\begin{aligned}
p&=\frac{1}{2c}\sqrt{2s\,2(s-a)\,2(s-b)\,2(s-c)}\\
&=\frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}
\end{aligned}$$

It is usual to denote the area of the triangle ABC by the Greek letter Δ

From Prop 21, Cor. 2, we have

$$\Delta=\frac{1}{2}pc$$

Hence $$\Delta=\sqrt{s(s-a)(s-b)(s-c)}$$

This is called Hero's Formula.

6 In the triangle ABC if p_1, p_2, p_3 be the perpendiculars from the

angular points A, B, C respectively on the opposite sides, then prove that

$$p_1=\frac{2\Delta}{a},\ p_2=\frac{2\Delta}{b},\ p_3=\frac{2\Delta}{c}$$

7 In the figure of Prop 28 prove that

$$AL=-\frac{b^2+c^2-a^2}{2c}\ \text{and}\ BL=\frac{c^2-a^2-b^2}{2c}$$

Show that by using this value of AL we get the same expressions for p and Δ as in Ex 6

8 Draw a triangle with sides 8 4 cm, 1 3 cm, and 8 5 cm, find the length of the perpendicular from the opposite angle on the longest side, and calculate the lengths of the segments of the longest side made by the perpendicular

Find also the area of the triangle

9 If m_1, m_2, m_3 be the medians which bisect the sides a, b, c of the triangle ABC, prove that

$$m_1^2=\frac{2b^2+2c^2-a^2}{4},\ m_2^2=\frac{2c^2+2a^2-b^2}{4},\ m_3^2=\frac{2a^2-2b^2-c^2}{4}.$$

• 10 Prove that four times the sum of the squares of the medians of a triangle is equal to three times the sum of the squares of its sides.

11 The sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of its four sides

MISCELLANEOUS QUESTIONS AND EXERCISES —II

1. Mention the first proposition in which two areas, which cannot be superposed, are proved equal to one another

2 Show how to cut the paper model of a parallelogram, so that when the parts are properly arranged they shall form a rectangle

3 Divide a parallelogram into two equal parallelograms

4 Define the *unit of area*

Construct a rectangle with sides 3" and 5"; divide adjacent sides into inches, and through the points of division draw parallels to the sides, show that the fifteen figures into which the rectangle is divided are squares with 1" side

5 Prove, by means of a construction, that if the sides of a rectangle are m and n inches respectively, its area contains mn square inches

6 Draw a parallelogram with sides 5" and 3 5", and included angle of 30°, construct a rectangle of equal area

Find also the area of the parallelogram

7 Draw a parallelogram with sides 5 cm and 8 cm, and included angle of 45°, construct another parallelogram of equal area, but having one of its angles of 60°

8 On a base 4.8" long draw two parallelograms of the same height 3", but having different angles of 150° and 60°

In the figure so formed prove that the areas of the parallelograms are equal

9 A point O is taken anywhere within the parallelogram $ABCD$, prove that the sum of the areas of the triangles OAB, OCD is constant

10 A triangle and a parallelogram stand on the same base, and the altitude of the triangle is double that of the parallelogram, prove that their areas are equal

11 The side AB of the parallelogram $ABCD$ is bisected in M. CM and DA produced meet in O. Prove that the triangle OCD is equal in area to the parallelogram

12 Two triangles PBC, QBC which are equal in all respects stand on the same base BC and on the same side of it, but are not coincident, BP, CQ being produced meet in R, prove that RP and RQ are equal

13 The sum of the perpendiculars drawn from any point within an equilateral triangle to its sides is equal to the altitude of the triangle.

14 The sum of the perpendiculars drawn from any point within a regular polygon to its sides is the same wherever the point is taken

15 The perpendiculars from two opposite angular points of a parallelogram, on the diagonal which does not pass through them, are equal

16 In the parallelogram $ABCD$, take any point P in the diagonal AC and join PB, PD, prove that the triangles APB, APD are equal in area

17 If a point O be taken within the parallelogram $ABCD$, and joined to the angular points, then

$$\Delta OAB + \Delta OCD = \Delta OBC + \Delta OAD$$

18 The sides of a triangle are 5", 6", and 7", calculate the length of the median which bisects the longest side.

19 The sides of a triangle are represented by the numbers 257, 255, and 32, calculate the length of the median which meets the longest side

20 The base of a triangle is 6", and its altitude is 4", find its area.

21 In the last exercise, if the triangle be isosceles, find the length of one of its equal sides

22 If a quadrilateral be bisected by each of its diagonals it is a parallelogram

23 Four times the sum of the squares of the medians of a right-angled triangle is equal to six times the square on the hypotenuse

24 $ABCD$ is a trapezium in which the sides AD and BC are parallel Produce AD, BC to E and F respectively, making DE and CF equal to BC and AD join EF Prove that $AEFB$ is a parallelogram double of the given trapezium

Hence deduce a rule for finding the area of a trapezium

25 The squares on the diagonals of a quadrilateral are together equal to twice the sum of the squares on the joins of the middle points of opposite sides

26 Prove that the triangle whose sides are $\frac{1}{2}(m+n)$, $\frac{1}{2}(m-n)$, $\sqrt{mn}$ is right angled

27 In a right-angled triangle the sides containing the right angle are 28 feet and 195 feet; find the hypotenuse.

28 The hypotenuse of a right-angled triangle is 101, and the shortest side is 20, find the third side

29 The hypotenuse of a right-angled triangle is 13 cm, and one of the sides is 6 5 cm, find the angles

30 If m and n be any numbers, the triangle whose sides are m^2+n^2, m^2-n^2, and $2mn$ is right-angled

31 Draw a triangle with sides 2 3″, 2 7″, and 3″, on the shortest side construct an isosceles triangle equal to it in area.

32 Draw a parallelogram with sides 5 cm and 6 8 cm, and included angle of 45°, construct a rhombus of equivalent area having the longest side of the parallelogram for one diagonal

33 Through the angular points of a triangle parallels are drawn to the opposite sides, prove that the triangle formed by the parallels is four times the original triangle

34 Draw a triangle with sides 2 9″, 3 2″, 3 5″, on the longest side as base construct a right angled triangle of equivalent area.

35 From any point, in the base of an isosceles triangle, perpendiculars are drawn to the other two sides, prove that the sum of the perpendiculars is the same wherever the point is taken within the base

36 The base of an isosceles triangle is 3 2 cm, and its perimeter is 16 2 cm, construct it and find the altitude

37 Each side of a rhombus is 5″ long, and one of its angles is of 60°; construct it and inscribe a square in it

38 *If D be the middle point of AB, and C any other point in AB or AB produced, prove that*

$$AB^2+AC^2=2AD^2+2CD^2$$

39 The parallel sides of a trapezium are 11 cm and 23 cm, and the other two sides are each equal to 10 cm; find the area

40 The base of a triangle is a, and the perpendicular on it from the opposite angle is p; show that the area is given by

$$\Delta = \tfrac{1}{2}ap$$

41 The diagonal of a quadrilateral is d, and the perpendiculars on it from the outlying angular points are p_1 and p_2; find the area.

42 The diagonals d, d' of a quadrilateral are at right angles to one another; find the area

43 The diagonals d, d' of a quadrilateral intersect at an angle of 30°; find the area

44 Prove that any straight line is equal to its own projection on a parallel straight line

45 The projections of two equal and parallel straight lines on any other straight line are equal

46 If the square on one side of a triangle be greater than the sum of the squares on the other two sides, the angle between these sides is obtuse.

47 If the square on one side of a triangle be less than the sum of the squares on the other two sides, the angle between these sides is acute

48 One inch being the unit of length, construct the line whose length is $\sqrt{5}$ inches

49 Prove that the triangle whose sides are 9.7 cm, 10 cm, and 10.4 cm is an acute angled triangle

50 Prove the corollary to Prop 29 when the perpendicular falls without the base

51 With sides 7 cm and 9 cm, and included angle of 60°, draw a parallelogram; construct a triangle equal to it and having one of its angles of 45°

52 The sides AB, AC of a triangle are bisected in F, E; D is any point in the base BC or BC produced; prove that the area of the triangle DEF is one-fourth of the area of ABC

53 Draw a parallelogram whose diagonals are 2″ and 2.5″, and one side is 1.3″; construct a rectangle of equal area, having the shortest side for base

54 Of two medians of a triangle, that is shorter which bisects the longer side

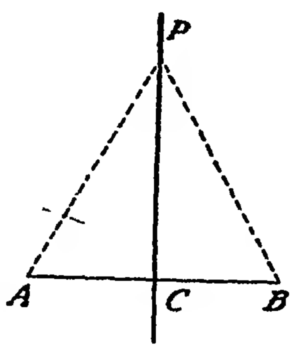
55 In any triangle the shortest median bisects the longest side.

LOCI

DEF.—*If any and every point on a line, part of a line, or group of lines, straight or curved, satisfies an assigned condition, and no other point does so, then that line, part of a line, or group of lines is called the locus of the point satisfying that condition*—(*Syllabus of the A I G T*)

PROPOSITION 30—THEOREM

The locus of a point which is equidistant from two fixed points is the perpendicular bisector of the straight line joining the two fixed points



Let A, B be two fixed points

Join AB and bisect it in C

$$CA = CB,$$

C is one point on the locus

Let P be any other point such that $PA = PB$ Join PC

In the Δs PAC, PBC,

$$\begin{cases} PA = PB, & [Hyp \\ CA = CB, & [Const. \\ \text{and } PC = PC, & \end{cases}$$

$\therefore$ Δs are congruent, [*Prop* 13.

and $\angle PCA = \angle PCB$,

each of these angles is a right angle

Hence every point which is equidistant from A and B lies on CP, the perpendicular bisector of AB

Moreover, if P be any point on CP, the perpendicular bisector of AB, then

$$PA = PB$$

For, in $\triangle$s PCA, PCB,

$$\left\{\begin{array}{ll} CA = CB, & [Hyp. \\ PC = PC, & \\ \angle PCA = \angle PCB, & [Hyp \end{array}\right.$$

$\triangle$s are congruent [*Prop* 9

Hence $PA = PB$

Note.—The student will notice that in the above proof we have demonstrated the following *two* theorems —

(i) *If a point satisfies the assigned condition, it is upon the locus*

(ii) *If a point is upon the locus, it satisfies the assigned condition*

In all similar cases both these associated theorems ought to be proved

EXERCISES

1 Find the locus of all points at a given distance from a fixed point

2 Find the locus of all points situated at a given distance from a given straight line

3 Find the locus of the centres of circles which pass through two fixed points

4 Find the locus of the vertices of isosceles triangles which have a common base

5 Find the locus of the centres of all circles which pass through a given point and have their radii equal to a given length

6 Find the locus of the vertices of all triangles of given area which stand on a common base

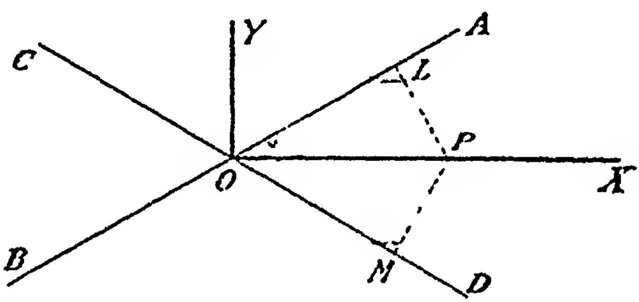
Proposition 31 —Theorem

The locus of a point which is equidistant from two intersecting straight lines consists of the pair of straight lines which bisect the angles between the two given lines

Let AOB, COD be the given st lines

Draw OX, OY bisecting the angles AOD, AOC respectively

Take any point P on OX, and draw the $\perp$rs PL, PM to the lines OA, OD.



Then, in Δs OPL, OPM,

$$\because \begin{cases} \angle POL = \angle POM, & [\textit{Const} \\ \angle PLO = \angle PMO, & [\textit{rt } \angle s \\ \text{and} \quad OP = OP, & \end{cases}$$

$\therefore$ Δs are congruent, [*Prop* 10

and $PL = PM$

Thus the perpendiculars from any point on the bisector OX to the lines OA and OD are equal.

Moreover, let P be any point within the angle AOD such that the perpendiculars from P on OA, OD are equal

Then in the right-angled Δs POL, POM,

$$\because \begin{cases} \text{side } PL = \text{side } PM, & [\textit{Hyp} \\ \text{hypotenuse } OP = \text{hypotenuse } OP, & \end{cases}$$

$\triangle$s are congruent, [*Prop* 14.

and $\angle POL = \angle POM$

Hence P is on the bisector of one of the angles between the given lines

EXERCISES

1 The locus of points, equidistant from two given parallel lines, is a straight line parallel to both the lines, and lying midway between them

2 Find the locus of the vertices of all triangles which stand on the same side of a given base and have a given area

3 From a given point straight lines are drawn to meet a fixed straight line, find the locus of their middle points

4 Find the locus of the middle points of the sides of triangles, which stand on the same base and are between the same parallels

5 Find the locus of the intersections of the diagonals of parallelograms, which stand on the same base, and are between the same parallels

6 A series of triangles have the same base, and medians of the same length bisecting the common base, find the locus of their vertices

Intersection of Loci —Let it be required to determine a point which satisfies two given conditions A and B

Let X be the locus of the point when it satisfies condition A alone, and Y its locus when it satisfies condition B alone, then the point or points common to X and Y will satisfy both conditions A and B

7 Find a point equidistant from three given points, A, B, C

8 Find a point equidistant from three given straight lines

9 On a given base a construct an isosceles triangle of given area Δ

10 Find a point equidistant from two given points, and at a given distance from another fixed point

11 Find a point equidistant from two given lines, and at a given distance from another fixed point

12 Construct a triangle ABC, having given the base a, the area Δ, and the median m which bisects BC

THE CIRCLE

We shall recapitulate here briefly certain properties of the circle with which the student is already familiar

A circle is the locus of a point in a plane which moves in such a way that its distance from a fixed point is constant

Circles are equal when their radii are equal For, if circles with equal radii are placed with their centres coincident, their circumferences must also coincide, otherwise in one or other of these circumferences there would be points unequally distant from the centre

A point lies without, upon, or within the circumference of a circle according as its distance from the centre is greater than, equal to, or less than the radius

A straight line cannot cut a circle in more than two points (See Prop. 17, Ex 3)

A circle is symmetrical about a diameter For if it is folded about any diameter the two parts must fall exactly on one another, otherwise there would be points on the two semi-circumferences unequally distant from the centre.

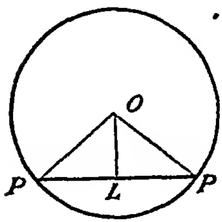
Proposition 32 —Theorem

A straight line, drawn from the centre of a circle to bisect a chord which is not a diameter, is at right angles to the chord

Conversely, the perpendicular to a chord from the centre bisects the chord

Let O be the centre of the ⊙, PP' a chord, and L the middle point of the chord; then

$$OL \text{ is } \perp PP'$$



Join OP, OP'

Then, in $\triangle$s OLP, OLP',

$$\begin{cases} PL = P'L, & [\textit{Hyp} \\ OP = OP', \\ OL = OL, \end{cases}$$

∴ $\triangle$s are congruent, [*Prop* 13

and $\angle OLP = \text{adj } \angle OLP'$,

∴ OL is $\perp PP'$

Next, let OL be drawn from the centre O perpendicular to any chord PP', then

$$PL = P'L$$

For, in the right-angled triangles OLP, OLP',

$$\begin{cases} \text{hypotenuse } OP = \text{hypotenuse } OP', \\ \text{side } OL = \text{side } OL, \end{cases}$$

∴ $\triangle$s are congruent, [*Prop* 14.

and $PL = P'L$

EXERCISES

1 *The perpendicular bisector of a chord passes through the centre*

2 *The locus of the centres of all circles which pass through two given points is the right bisector of the join of the points*

3 *The locus of the mid points of a set of parallel chords of a circle is the diameter perpendicular to them*

4 *Every diameter of a circle is an axis of symmetry*

5 The line joining the middle points of two parallel chords is perpendicular to both of them

6 If the line joining the middle points of two chords pass through the centre the chords are parallel

7 Given the middle point of a chord, draw it

8 Two concentric circles intercept between their circumferences equal portions of any straight line which cuts them both

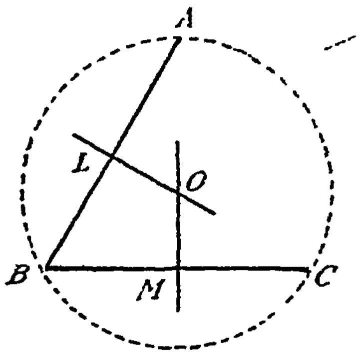
9 A parallelogram inscribed in a circle must be a rectangle

10 A trapezoid inscribed in a circle must have its non parallel sides equal

PROPOSITION 33 —THEOREM

There is one circle, and one only, which passes through three given points not in a straight line

Let A, B, C be three points not in a straight line then one circle, and one only, can be drawn to pass through them, *i e* one point, and one only, can be found which is equidistant from A, B, and C



Draw LO, MO the perpendicular bisectors of AB, BC, and let them intersect in O

$\because$ LO is the perpendicular bisector of AB,

$$OA = OB,$$ [*Prop* 30

and MO is the perpendicular bisector of BC,

$$OB = OC,$$ [*Prop* 30.

$$\therefore OA = OB = OC$$

Hence a circle whose centre is O and radius OA will pass through A, B, and C

Moreover, since the two lines LO and MO can intersect in one point only, therefore

O is the only point equidistant from A, B, and C Hence one circle only can be drawn through A, B, and C.

EXERCISES

1 *If two circles have three points common they coincide*

2 *If from any point within a circle three equal straight lines can be drawn to the circumference, that point must be the centre of the circle*

3 *Two circles cannot intersect in more than two points*

4 *Prove that the right bisectors of the sides of a triangle meet in the same point*

5 With sides 5 cm, 5 4 cm, and 6 3 cm draw a triangle and describe its circumscribing circle

6 Given the arc of a circle, find its centre and complete it

7 With radius 2″ describe a circle to pass through two points 2 4″ apart

8 Show that a circle can be described to pass through the angular points of any rectangle

9 If two sides of a quadrilateral are parallel, and the other two are equal, a circle can be described to pass through its angular points.

PROPOSITION 34 —THEOREM

In equal circles (or, in the same circle)—

(i) *If two arcs subtend equal angles at the centres, they are equal,*

(ii) *Conversely, if two arcs are equal, they subtend equal angles at the centres*

Let OP, CQ be two equal circles in which the arcs PP', QQ' subtend the angles POP', QCQ' respectively at the centres O and C

(i) If $\angle POP' = \angle QCQ'$,

then arc PP' = arc QQ'

Place the circle CQ on the circle OP, so that C falls on O, then

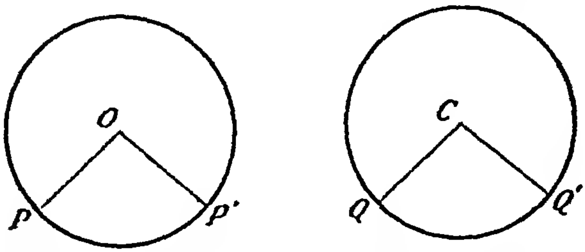
the circles are equal the circumferences will coincide

Now bring CQ into coincidence with OP, then

$$\because \angle QCQ' = \angle POP',$$

CQ' will coincide with OP';

$\therefore$ Q, Q' coincide with P, P' respectively



Hence the arc QQ' coincides with the arc PP',

$$\therefore \text{arc } PP' = \text{arc } QQ'.$$

(ii) If $\qquad$ arc PP' = arc QQ',

then $\qquad \angle POP' = \angle QCQ'$

Place the circle CQ on the circle OP, so that C falls on O; then

$\because$ the circles are equal the circumferences will coincide.

Again bring CQ into coincidence with OP, then

$\because$ arc PP' = arc QQ',

$\therefore$ Q' will coincide with P',

$\therefore$ CQ', OP' will coincide.

Hence $\qquad \angle POP' = \angle QCQ'$

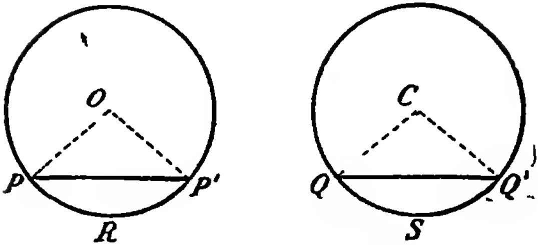
Proposition 35—Theorem

In equal circles (or, in the same circle)—

(i) *If two chords are equal, they cut off equal arcs;*

(ii) *Conversely, if two arcs are equal, the chords of the arcs are equal*

Let PRP', QSQ' be two equal circles whose centres are O and C respectively.



(i) If chord $PP' =$ chord QQ',
then arc $PRP' =$ arc QSQ'.

For, in Δs OPP', CQQ',

$$\begin{cases} OP = CQ, & \text{[Hyp.} \\ OP' = CQ', & \text{[Hyp.} \\ PP' = QQ', & \text{[Hyp.} \end{cases}$$

Δs are congruent, [Prop. 13

and $\angle POP' = \angle QCQ$

Hence arc $PRP' =$ arc QSQ' [Prop 34

(ii) If arc $PRP' =$ arc QSQ',
then chord $PP' =$ chord QQ'

arc $PRP' =$ arc QSQ',

$\angle POP' = \angle QCQ'$ [Prop 34

Again, in Δs OPP', CQQ',

$$\begin{cases} OP = CQ, & \text{[Hyp} \\ OP' = CQ', & \text{[Hyp} \\ \angle POP' = \angle QCQ', & \text{[Proved} \end{cases}$$

. Δs are congruent, [Prop 13

and $PP' = QQ'$.

EXERCISES

1 Prove this proposition, after the manner of Prop 34, by superposition

2 *In equal circles, if two chords are equal, they cut off equal segments*

3. *In equal circles, if two arcs are equal, the sectors contained by them are also equal.*

4. Triangles are inscribed in two equal circles such that two sides of the one are equal to two sides of the other, each to each; prove that their remaining sides are also equal.

5. *ABC* is the segment of a circle of which *AC* is the chord; prove that the right-bisector of *AC* bisects the arc *ABC*.

6. Prove that two parallel chords of a circle intercept equal arcs.

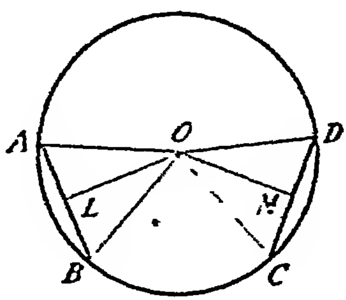
Proposition 36—Theorem

(i) *Equal chords of a circle are equidistant from the centre,*

(ii) *Conversely, chords of a circle that are equidistant from the centre are equal.*

Let AB, CD be chords of a circle whose centre is O.

Draw OL, $OM \perp AB$, CD respectively, and consequently bisecting AB, CD in L, M. [*Prop.* 32



(i) If $AB = CD$,

then $OL = OM$.

In the right-angled $\triangle$s OAL, OCM,

$\because$ $\begin{cases} \text{hypotenuse } OA = \text{hypotenuse } OC & [\textit{Def.} \\ \text{side } AL = \text{side } CM, & [\textit{Halves of equal st.} \end{cases}$

$\therefore$ $\triangle$s are congruent, [*Prop.* 1–

and $OL = OM$.

(ii) If $OL = OM$,

then $AB = CD$

In the right-angled Δs OAL, OCM,

$$\begin{cases} \text{hypotenuse } OA = \text{hypotenuse } OC, & \text{[Def.} \\ OL = OM, & \text{[Hyp.} \end{cases}$$

Δs are congruent, [*Prop.* 14

and $AL = CM$,

$AB = CD$ [*Doubles of equals.*

EXERCISES

1 *The locus of the middle points of all equal chords of a circle is a concentric circle*

2 *Of any two chords of a circle the one which is the greater is the nearer to the centre*

3 *Of any two chords of a circle the one which is the nearer to the centre is the greater*

4 If two chords of a circle intersect one another, and be equally inclined to the diameter through their point of intersection, they are equal

5 Two equal chords are placed in a circle, end to end, show that they are equally inclined to the diameter through their common point

6 A chord whose length is 70″ is placed in a circle of radius 37″, find the distance of the chord from the centre

7 The length of a chord is 30″, and its distance from the centre is 8″, find the radius of the circle

8 In a circle of radius 113 feet, a chord is drawn at a distance of 15 feet from the centre, find the length of the chord

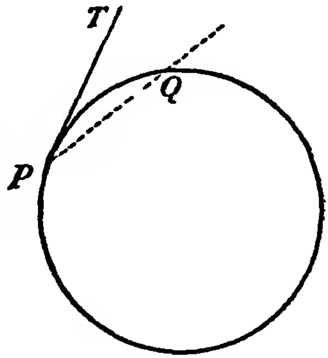
9 Draw a circle of 4 1″ radius, and place in it a chord 8″ long; calculate, and check by measurement, the distance of the chord from the centre of the circle

10 If two equal chords intersect within a circle, their segments are equal, the greater to the greater, and the less to the less

DEF —*Any straight line which cuts a circle is called a* **secant.**

DEF —*Let a secant cut a circle in two points P, Q;*

and let Q, travelling along the circle, approach indefinitely near to P, then the ultimate position (PT) of the secant PQ is called the tangent at P



The tangent therefore is a straight line which cuts a circle in two points which are so near to each other that they coalesce into one

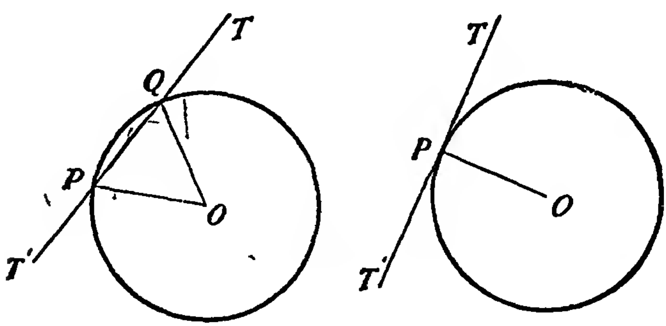
Hall 115.

PROPOSITION 37 —THEOREM

The tangent at any point of a circle and the radius through the point are perpendicular to one another.

Let P be any point on a circle whose centre is O, and let PT be the tangent at P, then

$$OP \perp PT$$



Let $T'T$ be a secant passing through P and cutting the circle again in Q

$$\because OP = OQ,$$

$$\angle OPQ = \angle OQP, \qquad \text{[Prop 11.}$$

$$\therefore \angle OPT' = \angle OQT. \qquad \text{[Supplements of equals.}$$

Now, whatever be the interval PQ this result is always true, therefore it is true when Q approaches P and ultimately coincides with it

In the last case PT becomes the tangent at P, and

$$\angle OPT' = \angle OPT.$$

Therefore $OP \perp TT'$. [Def

EXERCISES

1 At any point on the circumference of a circle, one, and only one, tangent can be drawn

2 The perpendicular to a tangent, at the point of contact, passes through the centre

3 Tangents at the extremities of a diameter of a circle are parallel

4 Find the locus of the centres of circles which touch a given straight line at a given point

5 Two circles are concentric, and a chord of the outer touches the inner, show that it is bisected at the point of contact

6 A series of equal chords are placed in a circle, prove that they are all tangents to a concentric circle

7 Tangents are drawn at the extremities of a chord of a circle, show that they make equal angles with the chord

8 Draw a tangent to a circle parallel to a given line

9 Draw a tangent to a circle perpendicular to a given line

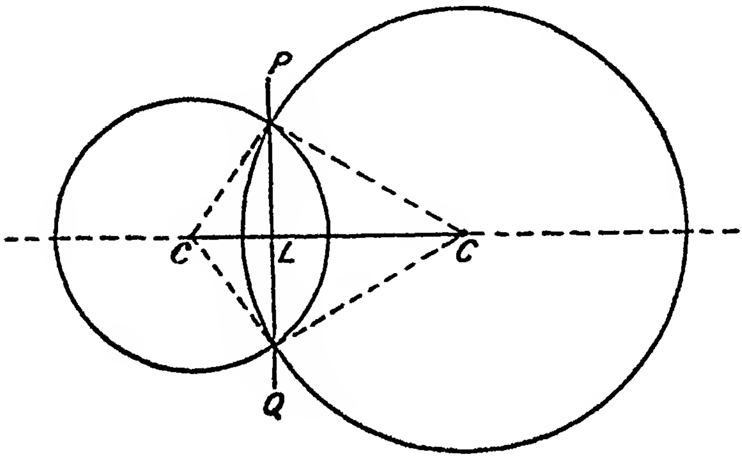
10 Draw a tangent to a circle making an angle of 60° with a fixed diameter

Touching Circles—Draw two circles cutting one another as in the figure

The line Cc, which passes through the centres of both circles, is called the **line of centres**

The line PQ, which joins the points of intersection, is called the **chord of intersection**, and when produced both ways it is called the **secant of intersection.** Each

circle is symmetrical about a diameter, hence it follows that the whole figure is symmetrical about the line of centres, so that if the figure is folded about the line *Cc* one part exactly fits the other.

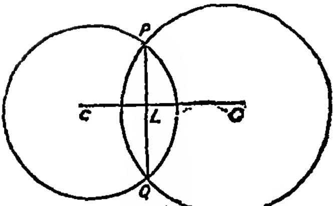


In the folded figure the point *P* will fall on the point *Q*; it follows therefore that—

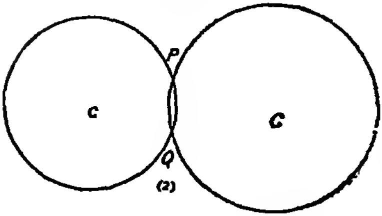
The line of centres is the perpendicular bisector of the chord of intersection.

Draw two circles, and cut them out carefully, so that the circumferences are well defined. Place one on top of the other as in (1).

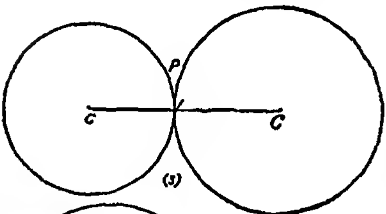
Now, keeping the small circle fixed, move the larger one towards the right into the position (2). During this operation the points *P* and *Q* will be coming nearer and nearer to each other and the line of centres, and since *PQ* is always bisected by the line of centres, they will arrive together at that line as in (3), and coalesce into one point *P*.



(1)



(2)



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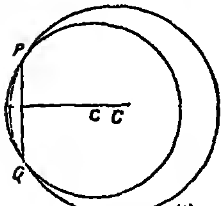
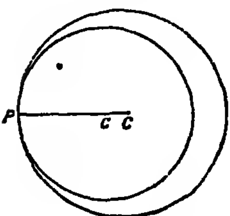


(4)



(5)

Similarly, if the larger circle is moved towards the left into the position (4), the points P and Q will approach each other, and finally coincide, as in (5), on the line of centres

When the points of intersection of two circles approach one another and coincide, the circles are said to touch one another

In (3) the circles touch one another *externally*, and the centres are as far apart as possible

In (5) the circles are touching internally, and the centres are as near each other as possible.

The point P in both cases is the point of contact.

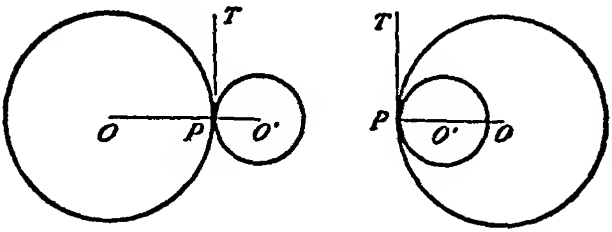
When the points of intersection of two circles coalesce, the secant of intersection PQ becomes the common tangent to the two circles at their point of contact.

Proposition 38—Theorem

If two circles touch, the point of contact lies on the straight line through the centres

Let two circles whose centres are O and O' touch at P, then

O, P, O' are in the same straight line.



Let PT be the common tangent to the circles at P, draw the radii OP, $O'P$ through the point of contact, then

$\angle$s OPT, $O'PT$ are rt $\angle$s, [*Prop* 37

$\therefore$ O, P, O' are in the same st. line [*Prop* 2.

EXERCISES

1 *If two circles touch one another externally, the distance between their centres is equal to the sum of their radii*

2 With radii $1\frac{3}{4}''$ and $1\frac{1}{4}''$ describe two circles touching one another externally

3 *If two circles touch one another internally, the distance between their centres is equal to the difference of their radii*

4 With radii 3 cm and 5 cm describe two circles touching one another internally

5 With radii of 1, 2, and 3 inches describe three circles, each touching the other two externally

6 What is the locus of the centres of circles of radius r which touch externally a fixed circle of radius R?

7 What is the locus of centres of circles of radius r which touch internally a fixed circle of radius R?

8 Find the locus of the centres of circles which touch a given circle at a given point

9 Two circles of radii r_1 and r_2 touch one another externally; describe a third circle of radius r_3 to touch the other two (i) externally and (ii) internally

10 Two circles of radii 2″ and 3″ touch one another externally; describe a third circle of radius 5″ to be touched by both internally

11 Find the locus of the centres of circles which touch a given straight line at a given point

12 Describe a circle which touches a given straight line at a given point and has its centre on another given straight line

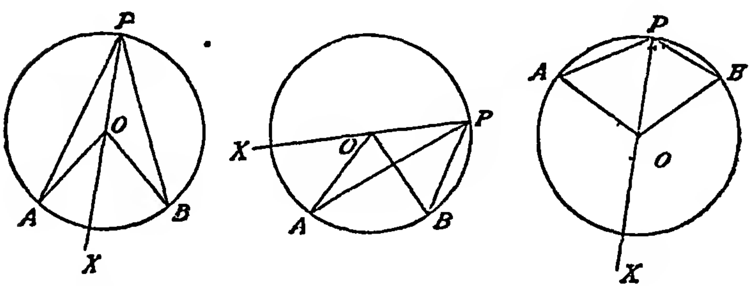
13 Describe a circle which touches a given circle at a given point, and has its centre on another given circle

Proposition 39—Theorem

The angle which an arc of a circle subtends at the centre is double that which it subtends at any point on the remaining part of the circumference

Let AB be an arc of a circle whose centre is O, and let P be any point on the remaining part of the circumference, then

$$\angle AOB = 2\angle APB$$



Join PO and produce it to X

Then $\because AO = PO$,

$\angle OAP = \angle OPA$ [*Prop.* 11.

But ext. $\angle AOX = \angle OAP + \angle OPA$; [*Prop* 7, *Cor.*

$\angle AOX = 2\angle OPA$

Similarly, $\angle BOX = 2\angle OPB$

Hence $\angle BOX \pm \angle AOX = 2(\angle OPB \pm \angle OPA)$.

For the first figure take the upper sign, then

$$\angle AOB = 2\angle APB.$$

For the second figure take the lower sign, then

$$\angle AOB = 2\angle APB.$$

For the third figure take the upper sign, then

$$\angle \widehat{AOB} = 2\angle APB$$

Note.—The symbol $\frown$ placed over $\angle AOB$ indicates that, of the two angles made by OA and OB at the point O, the one which is *greater than two right angles* is to be taken

An angle greater than two right angles is sometimes called a **reflex angle.**

The reflex angle AOB in the third figure is marked with a dotted arc.

PROPOSITION 40.—THEOREM

(i) *Angles in the same segment of a circle are equal,*

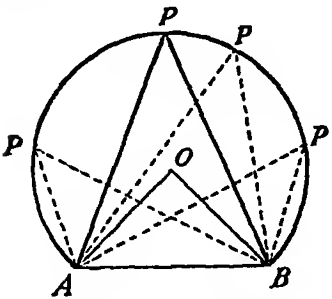
and (ii) *If the line joining two points subtends equal angles at two other points on the same side of it, the four points lie on a circle.*

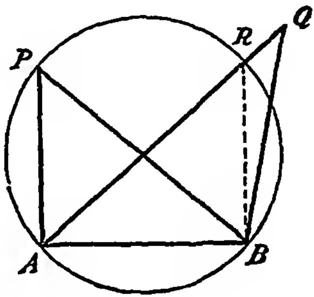
(i) Let APB be a segment of a circle whose centre is O, and P any point on the arc of the segment, then for all positions of P on the arc the magnitude of the angle APB is the same.

For, wherever be the position of P,

$$\angle APB = \tfrac{1}{2}\angle AOB \qquad \text{[Prop 39}$$

Hence $\angle APB$ is constant for all positions of P on the arc.





(ii) Let A, B and P, Q be four points such that

$$\angle APB = \angle AQB,$$

then $\qquad A, B, P, Q$ lie on a circle

A circle can be described to pass through APB

[*Prop* 33.

If this circle does not pass through Q, let it cut AQ, or AQ produced, in R

Join BR

Then $\qquad \angle$s APB, ARB are in the same segment,

$$\angle APB = \angle ARB$$

But $\qquad \angle APB = \angle AQB \qquad$ [*Hyp.*

$$\angle ARB = \angle AQB$$

But this is impossible, for one of these angles is an exterior angle of the triangle BRQ, and the other is an interior opposite angle [*Prop* 7, *Cor*

Hence the circle through A, B, P must pass through Q.

Def —Four or more points which are such that a circle can be made to pass through them are said to be **concyclic.**

DEF.—A quadrilateral which can be inscribed in a circle is called a **cyclic quadrilateral**.

EXERCISES

1. *If a given straight line subtend equal angles at any number of points on the same side of it, all these points are concyclic.*

2. *The locus of a point on one side of a given straight line at which that line subtends a constant angle is an arc of which that line is the chord.*

3. *The base and vertical angle of a triangle being given, the locus of the vertex is a segment of a circle.*

4. *The internal bisector of an angle at the circumference of a circle bisects the arc on which it stands, and the external bisector bisects the arc of the segment which contains the angle.*

5. Given the base and vertical angle of a triangle, prove that the internal and external bisectors of the vertical angle pass through fixed points.

6. *AOB*, *COD* are two intersecting chords of a circle; prove that the triangles *OAC*, *OBD* are equiangular.

7. Given three points on the circumference of a circle, find a fourth without finding the centre.

8. In an acute angled triangle perpendiculars are drawn from two of the angular points on the opposite sides; prove that the feet of the perpendiculars and these two angular points are concyclic.

9. *What does this proposition become when P coincides with B or A?*

PROPOSITION 41.—THEOREM

(i.) *The angle in a semicircle is a right angle;*

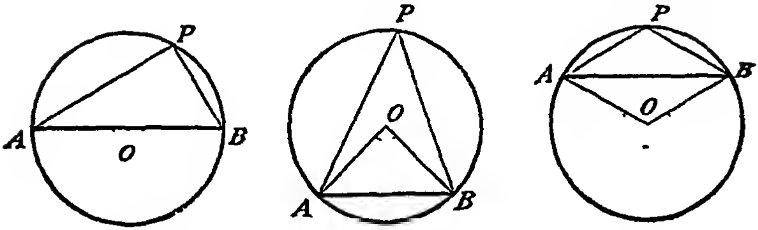
(ii.) *The angle in a segment greater than a semicircle is less than a right angle,*

and (iii.) *The angle in a segment less than a semicircle is greater than a right angle.*

Let *APB* be the segment of a circle whose centre is *O*. Let *APB*, the angle in the segment, be denoted by *P*, and *AOB*, the corresponding angle at the centre, be denoted by *C*.

Then, whatever be the size of the segment,

$$\angle P = \tfrac{1}{2}\angle O \qquad \text{[Prop 39}$$



(i) When the segment is a semicircle,

$$\angle O = 2 \text{ rt } \angle s,$$
$$\therefore \angle P = \text{a rt } \angle.$$

(ii) When the segment is greater than a semicircle,

$$\angle O < 2 \text{ rt } \angle s,$$
$$\therefore \angle P < \text{a rt } \angle.$$

(iii) When the segment is less than a semicircle,

$$\angle \hat{O} > 2 \text{ rt } \angle s,$$
$$\therefore \angle P > \text{a rt } \angle$$

Cor —*The circle described on the hypotenuse of a right-angled triangle as diameter, passes through the right angle*

Proposition 42.—Theorem

(i) *The opposite angles of any quadrilateral inscribed in a circle are supplementary,*

(ii) *Conversely, if the opposite angles of a quadrilateral are supplementary it can be inscribed in a circle*

(i) Let $ABCD$ be a quadrilateral inscribed in a circle whose centre is O, then

$$\angle s\ A,\ C \text{ and } \angle s\ B,\ D \text{ are supplementary.}$$

For $\qquad \angle\beta = 2\angle B,$

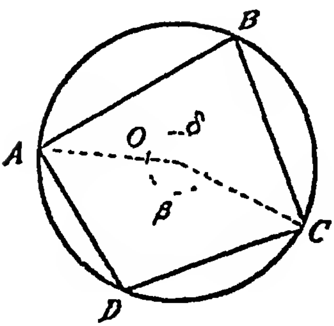
and $\qquad \angle\hat{\delta} = 2\angle D,$ [*Prop* 39

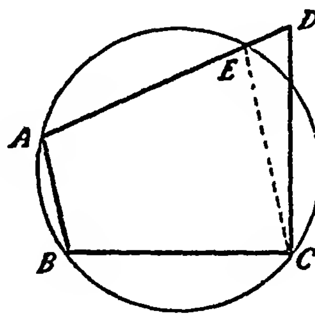
$\therefore \angle\beta + \angle\hat{\delta} = 2\angle B + 2\angle D$

But $\qquad \angle\beta + \angle\hat{\delta} = 4$ rt $\angle$s, [*Prop* 1, *Cor*

$\therefore \angle B + \angle D = 2$ rt $\angle$s

Similarly, $\qquad \angle A + \angle C = 2$ rt $\angle$s





(ii) Let $ABCD$ be a quadrilateral in which the angles B, D are supplementary, then a circle can be described to pass through A, B, C, D

A circle can be described to pass through A, B, C [*Prop.* 33

If this circle does not pass through D it will cut AD, or AD produced, in some point E

Join CE

Then $\qquad \therefore ABCE$ is a cyclic quadrilateral,

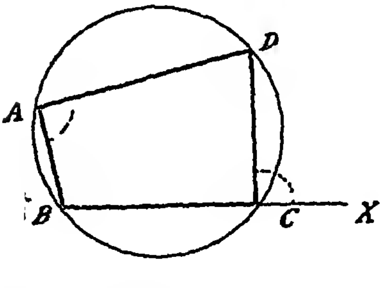
$\angle ABC + \angle CEA = 2$ rt $\angle$s [*Proved*

But $\qquad \angle ABC + \angle EDC = 2$ rt $\angle$s, [*Hyp*

$\therefore \angle CEA = \angle EDC,$

but this is impossible, for one of these angles is an exterior angle of the triangle EDC, and the other is an interior opposite angle.

Hence the circle through A, B, C must pass through D



COR 1—*If a side of a cyclic quadrilateral be produced, the exterior angle is equal to the interior opposite angle*

$(\angle DCX = \angle BAD)$

COR 2—*If, when one side of a quadrilateral is produced, the exterior angle is equal to the interior opposite angle, the quadrilateral is cyclic*

EXERCISES

1 Circles are described on two adjacent sides of a triangle as diameters, prove that they intersect the third side in the same point

2 The perpendiculars AD, BE to the sides BC, CA of the triangle ABC intersect in P; prove that $PDCE$ is a cyclic quadrilateral

3 The sides AB, CD of a cyclic quadrilateral $ABCD$ are produced to meet in O, prove that OAC, OBD are equiangular triangles.

4. A triangle is inscribed in a circle, prove that the sum of the angles in the exterior segments cut off by the sides is equal to four right angles

5 Every rectangle is a cyclic quadrilateral

6 The sum of the three alternate angles of a cyclic irregular hexagon is equal to four right angles

7 In a cyclic hexagon two pairs of opposite sides are respectively parallel to each other, prove that the remaining pair of sides are also parallel

8 The non parallel sides of a cyclic trapezium are equal

9 The circles described on the sides of a rhombus as diameters all pass through the same point

10 What does this proposition become if two angular points of the quadrilateral coincide?

11 The sum of the angles in the four segments of the circle exterior to a cyclic quadrilateral is equal to six right angles.

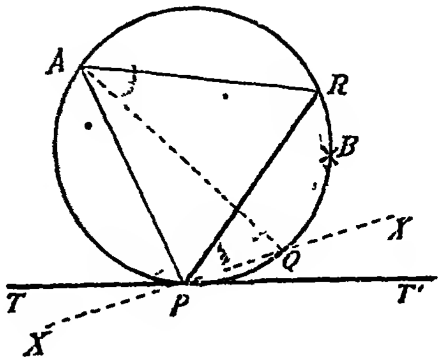
PROPOSITION 43—THEOREM

If a straight line touch a circle, and from the point of contact a chord be drawn, the angles which this chord makes with the tangent are equal to the angles in the alternate segments

Let the line TPT' touch the circle $PARB$ in the point P, let PR be a chord drawn through P, and let A and B be any two points on the segments made by PR Then

$$\angle RPT' = \angle RAP$$

and $\angle RPT = \angle RBP$.



Let XX' be any line passing through P and meeting the circle again in Q

Then whatever be the interval PQ,

$$\angle RPQ = \angle RAQ \qquad \text{[Prop 40.}$$

Now let the line XX' revolve round P, so that Q approaches P and ultimately coincides with it. Then XX' will coincide with the tangent TT',

$$\therefore \angle RPT' = \angle RAP$$

In the same manner we can prove that

$$\angle RPT = \angle RBP.$$

EXERCISES

1 *If through an extremity of a chord of a circle a straight line be drawn, such that either of the angles it makes with the chord is equal to the angle in the alternate segment, the straight line is a tangent to the circle*

2 The chord joining the points of contact of parallel tangents to a circle is a diameter

3 If two straight lines be drawn through the point of contact of two touching circles, the chords joining their extremities are parallel

4. Prove that the tangents at the extremities of the chords in Ex. 3 are parallel, two by two

5 If two circles touch internally, or externally, any line drawn through the point of contact will cut off similar segments

6 A straight line $B'C'$ is drawn parallel to the base BC of the triangle ABC, meeting the sides in B' and C' Prove that the circumcircles of the triangles ABC, $AB'C'$ touch one another at A

7 The diagonals of the parallelogram $ABCD$ intersect in E, prove that the circumcircles of the triangles EBC, EAD touch one another at E

Proposition 44—Theorem

If two chords of a circle intersect either inside or outside the circle the rectangle contained by the parts of the one is equal to the rectangle contained by the parts of the other

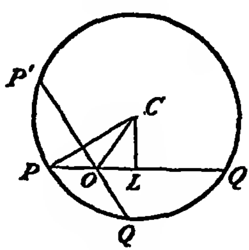
Let the chords PQ, $P'Q'$ of a circle, whose centre is C and radius r, intersect in the point O Then, whether O be inside or outside the circle,

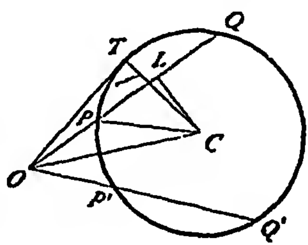
$$OP \cdot OQ = OP' \cdot OQ'.$$

Draw CL perpendicular to PQ, then

$$PL = QL \qquad [Prop\ 32$$

Join CP, CO





(1) When the point O is inside the circle,

$$OP \cdot OQ = (PL - OL)(PL + OL)$$
$$= PL^2 - OL^2 \qquad [Prop\ 26$$

$$= (PL^2 + CL^2) - (OL^2 + CL^2)$$
$$= r^2 - OC^2. \qquad [\text{Prop } 27$$

Similarly, $OP'.OQ' = r^2 - OC^2$

$$\therefore\ OP\ OQ = OP'.OQ'$$

(ii) When the point O is outside the circle, draw OT a tangent to the circle, and join CT Then

$$\angle OTC \text{ is a rt } \angle \qquad [\text{Prop } 37.$$

We have

$$OP\ OQ = (OL - PL)(OL + PL)$$
$$= OL^2 - PL^2 \qquad [\text{Prop. } 26.$$
$$= (OL^2 + CL^2) - (PL^2 + CL^2)$$
$$= OC^2 - r^2, \qquad [\text{Prop. } 27.$$
$$\therefore\ OP\ OQ = OC^2 - CT^2$$
$$= OT^2$$

Similarly, $OP'.OQ' = OT^2$,

$$\therefore\ OP\ OQ = OP'\ OQ'$$

COR 1 —*If from any point without a circle a secant and a tangent be drawn to the circle, the rectangle contained by the secant and its external segment is equal to the square on the tangent.*

$$(OP\ OQ = OT^2)$$

COR 2 —*If from any point without a circle two straight lines be drawn, one of which is a secant, and the other meets the circle, and if the rectangle contained by the secant and its external segment be equal to the square on the line which meets the circle, that line is a tangent to the circle.*

EXERCISES

1 *If two tangents to a circle be drawn from the same point, they are equal*

2 *If from any point, on the common chord of two intersecting circles, tangents can be drawn to the circles, these tangents are equal*

3 *If the tangents from any point to two intersecting circles are equal, the point lies on the common chord of the circles*

4. *The locus of points from which tangents drawn to two intersecting circles are equal consists of two portions of their common chord*

5 *The common chord of two intersecting circles bisects both their common tangents*

6 *The common chords of three intersecting circles, taken two and two, pass through the same point.*

7 *If from any point on the tangent to two touching circles at their point of contact tangents be drawn to the circles, these tangents are equal*

8 *If from any point on the common chord produced of two intersecting circles two secants be drawn, one to each circle, their four points of intersection are concyclic*

9 What is the area of the rectangle contained by the segments of a chord drawn through a point 2 4″ distant from the centre of a circle whose radius is 2 5″?

10 What is the area of the rectangle contained by the segments of a chord drawn through a point 1 ft 3 in distant from the centre of a circle of radius 1 ft 5 in ?

11 In a circle of radius r a chord is drawn through a point within the circle whose distance from the centre is c, show that the area of the rectangle contained by the segments of the chord is

$$r^2 - c^2.$$

12. The radii of two circles are 5″ and 12″, and the distance between their centres is 13″, find the length of their common chord

13 The radii of two circles are a and b, and the distance between their centres is c, show that the length of their common chord is

$$\frac{4\sqrt{s(s-a)(s-b)(s-c)}}{c}$$

where $$2s = a + b + c$$

MISCELLANEOUS QUESTIONS AND EXERCISES —III

1 Define a *circle*

Show that a circle is symmetrical about any diameter

2 Find the centre of a given circle, and show that a circle cannot have more than one centre.

3 Show that the right-bisectors of the sides of any rectilineal figure inscribed in a circle all pass through the same point.

4 The distance between the centres of two intersecting circles is less than the sum of their radii and greater than their difference

5 If through one of the points of intersection of two intersecting circles a straight line be drawn parallel to the line of centres, and terminated by the circumferences, its length is twice the distance between the centres

6 The line joining the middle points of any two parallel chords of a circle passes through the centre

7 Equal chords of a circle which are co terminous are equally inclined to the radius through their common point

8 Enunciate and prove the converse of Ex 7

9 Prove that if the radii of two circles are equal they are equal in every respect

10 Explain what is meant by the angle subtended by an arc (i) at the centre, (ii) at the circumference of a circle

11. A chord of constant length slides round the circumference of a given circle ; prove that

(i) the locus of its middle point is a circle ,

(ii) it touches a circle ,

and (iii) the locus of a point fixed in the chord is a circle

12 A chord of length $2c$ is drawn in a circle of radius r, find its distance from the centre.

13 Two equal chords are placed in a circle , prove that a concentric circle which cuts them will make equal intercepts on both

14. What is the shortest chord which can be drawn through a given point within a circle?

15 If chords of a circle be tangents to a concentric circle they are equal

16 TT' is a common tangent to two circles whose centres are O, O' respectively , prove that OT, $O'T'$ are parallel

17 If one circle lies entirely inside another without meeting it the difference between their radii is greater than the distance between their centres

18 If one circle lies entirely outside another without meeting it the sum of their radii must be less than the distance between their centres

19 With radii 2 3″ and 1 6″ draw two circles to touch one another externally

20 With radii 3 5 cm and 4.8 cm draw circles to touch one another internally

21 With radii 1 5″, [illegible]″, and 1 8″ draw three circles, each touching the other two externally

22 Prove that the chord of a circle drawn through the middle point of another chord cannot be lesser than that chord

23 Prove that the longest chord in a circle is a diameter

24 Draw two intersecting circles, radii 2 and 3 in, whose common chord is 1 in

25 With radii 1 8′ and 2 7″ draw two intersecting circles whose common chord is the greatest possible

26 Find the locus of the middle points of all chords of a circle passing through a point on its circumference

27 Find the locus of the middle points of all chords of a circle drawn through a given point within it

28 From a given point without a circle secants are drawn to the circle, find the locus of the middle points of the portions of the secants intercepted by the circle

29 From different points on the circumference of a circle tangents of equal lengths are drawn, find the locus of their extremities

30 Straight lines joining the opposite extremities of two parallel chords of a circle intersect on the diameter which bisects the chords

31 The radius of the circle circumscribing the triangle ABC is given, and the angle A is also given, show that the side opposite the angle A has a determined length

32 If from any point in a circular arc perpendiculars be let fall on the radii to its extremities, the line joining the feet of these perpendiculars will be of invariable length

33 If two chords of a circle bisect one another they must be diameters

34 Circles are described on the two equal sides of an isosceles triangle as diameters, prove that they intersect at the middle point of the third side.

35 In the base BC of the triangle ABC a point P is taken, such that the angle APC is equal to the angle BAC, show that CA touches the circle circumscribing the triangle ABP

36 If the diagonals of a trapezium are equal its angular points are concyclic

37 In a cyclic quadrilateral the bisectors of any angle, and of the opposite exterior angle, intersect on the circle circumscribing the quadrilateral

38 *One side of a triangle is equal to one side of another, and the angles opposite to these sides are supplementary, prove that the circles circumscribing the two triangles are equal*

39 In the triangle ABC the perpendiculars from B and C on the

opposite sides intersect in P; show that the circles circumscribing the triangles ABC, PBC are equal

40 In how many points can two circles cut one another?

Why cannot two touching circles have any other common point?

41. Define *angle in a segment* and *similar segments*

Give examples of angles which are greater than two right angles

42 A ladder gradually slips down a wall. find the locus of its middle point.

43 Draw a chord of a circle, whose length is equal to twice its distance from the centre.

44 Draw a chord of a circle, whose distance from the centre is equal to half the length of the radius.

45 The opposite sides of a cyclic quadrilateral are produced to meet Prove that the bisectors of the two angles so formed are at right angles to one another

46 A series of right-angled triangles are described on the same hypotenuse, find the locus of the vertices of the right angles

47 Find the locus of a point which moves so that the sum of the squares of its distances from two fixed points is constant

48 Through the point of contact of two touching circles a straight line is drawn to cut the circles again in two points, prove that the radii drawn to these points are parallel

49. If the diagonals of a cyclic quadrilateral are at right angles, the perpendicular from their intersection on any side, being produced, bisects the opposite side.—*Brahmagupta's Theorem*

PROPORTION

DEF —*When two quantities of the same kind are such that a unit can be found which measures each of them exactly, the two quantities are said to be* **commensurable**

Thus if A and B be two lengths, and u a unit of length such that

$$A = mu \text{ and } B = nu,$$

where m and n are integers, then A and B are said to be commensurable.

DEF —*Two quantities which are not commensurable are said to be* **incommensurable**

DEF —**The ratio of A to B** *is the number of times (whole or fractional) that B is contained in A*

Thus if A and B be two commensurable quantities, such that $A = mu$, $B = nu$, the result of division may be expressed as

$$\frac{A}{B} = \frac{m}{n}$$

Hence the ratio of two commensurable quantities can be expressed as a fraction whose numerator and denominator are integers In what follows we shall only deal with commensurable magnitudes

DEF.—*Four quantities A, B, C, D are said to be proportionals when*

$$\frac{A}{B}=\frac{C}{D}$$

That is, when the ratio of the first quantity to the second is equal to the ratio of the third to the fourth, the four quantities are proportionals

We expect the student to be already familiar with the properties of proportionals, such as the following:—

If $$\frac{A}{B}=\frac{C}{D},$$

then $$\frac{A}{A+B}=\frac{C}{C+D},$$

and we shall quote these without further proof.

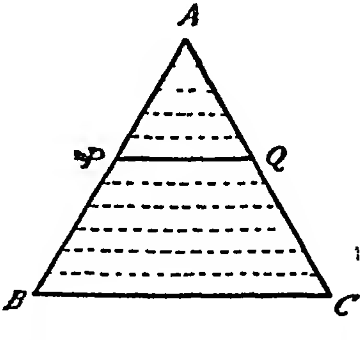
PROPOSITION 45—THEOREM

(i) *If a straight line is drawn parallel to one side of a triangle, the other two sides are divided proportionally*

(ii) *Conversely, if a straight line divides two sides of a triangle proportionally, it is parallel to the third side*

(i) Let PQ be a st. line drawn ‖ to BC, a side of the $\triangle ABC$, meeting AB, AC in P, Q respectively, then

$$\frac{AP}{PB}=\frac{AQ}{QC}$$



Let AB be divided into $m+n$ equal parts, of which AP contains m and PB contains n, then

$$\frac{AP}{PB}=\frac{m}{n}.$$

Through the points of division draw parallels to BC.

PQ will be one of these parallels

The parallels will divide AC into $m+n$ equal parts (Prop 19), AQ containing m and QC n

$$\frac{AQ}{QC}=\frac{m}{n},$$

$$\frac{AP}{PB}=\frac{AQ}{QC}$$

Cor.

$$\frac{AP}{AB}=\frac{AQ}{AC}$$

(ii) Let the line PQ meet the sides of $\triangle ABC$ in P, Q so that

$$\frac{AP}{PB}=\frac{AQ}{QC},$$

or, which is the same thing,

$$\frac{AP}{AB}=\frac{AQ}{AC},$$

then PQ is $\parallel$ to BC

For, if PQ is not $\parallel$ to BC, let PQ' be $\parallel$ to BC, then

$$\frac{AP}{AB}=\frac{AQ'}{AC} \qquad \text{[Part (i), Cor.}$$

But

$$\frac{AP}{AB}=\frac{AQ}{AC}, \qquad \text{[Hyp.}$$

$$\frac{AQ'}{AC}=\frac{AQ}{AC}$$

Hence Q, Q' are the same point.

Therefore $PQ \parallel BC$

EXERCISES

1 If through the middle point of a side of a triangle a straight line be drawn parallel to the base, it will bisect the other side

2 The straight line joining the middle points of two sides of a triangle is parallel to the third side.

3 In the figure of this proposition, if the parallel PQ cut AB, AC produced in P, Q respectively, then

$$\frac{AP}{PB}=\frac{AQ}{QC}, \text{ and the converse}$$

Show that the above proof may be read with the new figure

4 If the parallel PQ cut BA, CA produced through the vertex in P, Q respectively, then

$$\frac{AP}{PB}=\frac{AQ}{QC}, \text{ and the converse}$$

Show that the same proof applies to this figure also

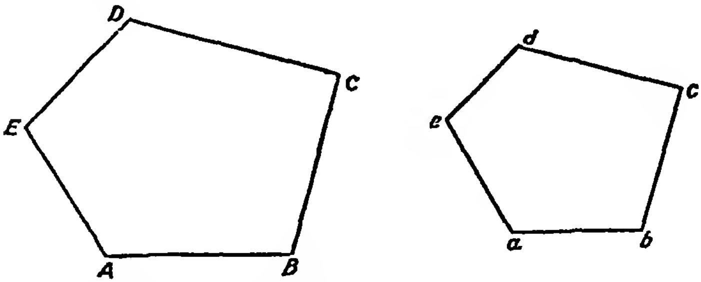
5 *The intercepts which three parallel straight lines make on any two straight lines are proportional*

6 A parallel to the parallel sides of a trapezium divides the non-parallel sides proportionally.

SIMILAR TRIANGLES

DEF —**Similar rectilineal figures** *are equiangular to one another, and their sides about the equal angles are proportional*

Let the two figures *ABCDE* and *abcde* be similar,



then the angles A, B, C, D, E taken in order are equal respectively to the angles a, b, c, d, e taken in order, and $\frac{AB}{ab}=\frac{BC}{bc}=\frac{CD}{cd}=\frac{DE}{de}=\frac{EA}{ea}$ Thus if the side AB be double of ab, then every side of the first polygon will be double of the corresponding side of the second polygon

Again, if AB ab 3 2,

then BC bc 3 2, and so on

We see, then, that *similar figures must have* (1) *their angles equal, and* (2) *the ratios of their corresponding sides, which lie between pairs of equal angles, also equal*

In what follows we shall show that in the case of a triangle when one condition of similarity is satisfied the other follows

DEF —*In equiangular triangles the sides which are opposite to the equal angles are called* **corresponding sides.**

Thus if ABC, PQR are two equiangular triangles, in which the angles A, B, C are respectively equal to the angles P, Q, R, then

$$(BC, QR), (CA, RP), \text{ and } (AB, PQ)$$

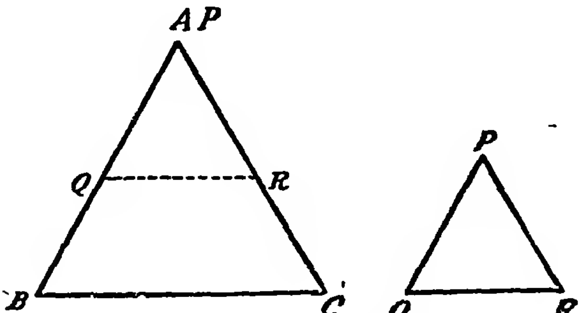
are pairs of corresponding sides.

PROPOSITION 46 —THEOREM

If two triangles are equiangular their corresponding sides are proportional

Let ABC, PQR be two $\triangle$s, such that $\angle$s A, B, C = $\angle$s P, Q, R respectively; then

$$\frac{PQ}{AB}=\frac{QR}{BC}=\frac{PR}{AC}$$



Place $\triangle PQR$ on $\triangle ABC$ so that the $\angle P$ falls on the equal $\angle A$, and the corresponding sides PQ, AB are in the same direction.

OT are drawn to the circle; prove that the triangles OAT, OBT are similar, and hence obtain the equation

$$OA \cdot OB = OT^2.$$

6 *In equiangular triangles the perpendiculars drawn from the angles on corresponding sides are proportional to those sides*

7 *All parallels drawn to the base of a triangle, which are terminated by the sides, are bisected by the median to that base*

8 *The perimeters of equiangular triangles are proportional to their corresponding sides*

9 On a base 3″ long draw a triangle ABC, having $\angle B = 60°$ and $AB = 2.5″$; from a point P in AB, distant 1″ from A, draw a parallel to BC and calculate its length

10 A man 6 ft in height has a shadow 11 ft in length, and at the same time the shadow of a tower is 143 ft in length; find the height of the tower

11 The sides containing the right angle of a right-angled triangle are 24 ft, 143 ft respectively, find the length of the perpendicular from the right angle on the hypotenuse.

PROPOSITION 47.—THEOREM

If the three sides of one triangle are proportional to the three sides of another the triangles are equiangular.

Let ABC, PQR be two Δs, such that

$$\frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR};$$

then the triangles are equiangular.

Let $\angle SQR = \angle B$ and $\angle SRQ = \angle C$, then Δs QRS, ABC are equiangular;

$$\therefore \frac{AB}{QS} = \frac{BC}{QR}. \qquad \text{[Prop. 46.}$$

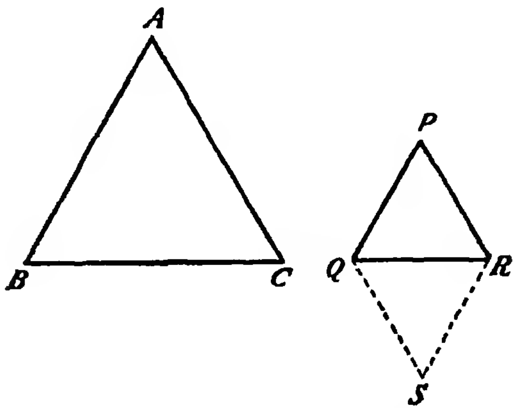
But
$$\frac{BC}{QR} = \frac{AB}{PQ}; \qquad \text{[Hyp.}$$

$$\therefore \frac{AB}{QS} = \frac{AB}{PQ},$$

$$\therefore QS = PQ.$$

Similarly, $SR = PR$

Also QR is common to the $\triangle$s PQR, SQR,



$\therefore$ $\triangle$s PQR, SQR are congruent and equiangular [*Prop* 13.

But $\triangle$s ABC, SQR are equiangular,

$\triangle$s ABC, PQR are equiangular

PROPOSITION 48 — THEOREM

If two triangles have one angle of the one equal to one angle of the other, and the sides about these equal angles proportional, the triangles are similar

Let ABC, PQR be two $\triangle$s such that

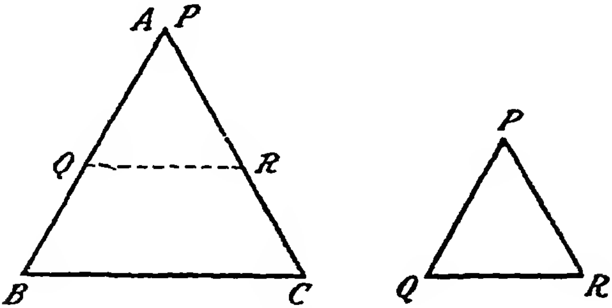
$$\angle P = \angle A$$

and
$$\frac{PQ}{AB} = \frac{PR}{AC},$$

then the $\triangle$s are similar

Place the $\triangle PQR$ on the $\triangle ABC$ so that P falls on

A and Q lies on AB, or AB produced; then since the $\angle$s P and A are equal, R will lie on AC



Now $\qquad \because \dfrac{PQ}{AB} = \dfrac{PR}{AC}$, [*Hyp.*

$\therefore\ QR \parallel BC$; [*Prop.* 45.

$\therefore\ \angle Q = \angle B$,

and $\qquad \angle R = \angle C$,

[*Corresponding angles.*

$\therefore$ Δs PQR, ABC are equiangular and similar.

EXERCISES

1. The sides of two triangles are 1.2″, 1.4″, 1.5″, and 1.8″, 2.1″, 2.25″ respectively, draw the triangles and prove that they are similar

2. Draw two triangles ABC, $A'B'C'$, having given $a=2''$, $b=1.6''$, $C=75°$, $a'=1.5''$, $b'=1.2''$, $C'=75°$. Prove that the triangles are similar

3. The joins of the middle points of the sides of a triangle form another triangle similar to the given triangle

4. In Ex 3 prove that the remaining three triangles are also similar to the given triangle

5. In the isosceles triangle ABC the base AB is produced both ways to P and Q, so that $AP.BQ=AC^2$; prove that the triangles PAC, QBC are similar

6. The perpendicular drawn from the vertex of a triangle on the

base is a mean proportional between the segments of the base, prove that the triangle is right-angled

7 The diagonals of a quadrilateral cut one another in the same ratio, prove that the quadrilateral has two sides parallel

8 Find the locus of a point which moves in such a way that of the perpendiculars drawn from it to the sides AB, AC of a triangle, the first is always twice as long as the second

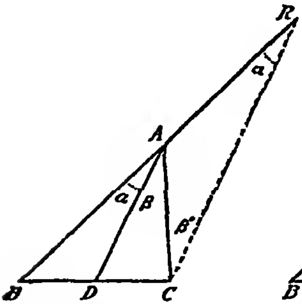
PROPOSITION 49 —THEOREM

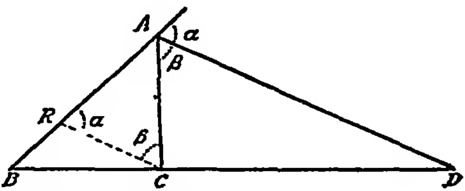
The internal bisector of an angle of a triangle divides the opposite side internally in the ratio of the sides containing the angle, and likewise the external bisector externally

Let AD be the bisector of the $\angle A$ of the $\triangle ABC$, then, according as AD is the internal or external bisector, it will meet BC or BC produced in some point D

In either case

$$\frac{BD}{DC}=\frac{AB}{AC}$$





Draw $CR \parallel DA$, cutting BA produced, or BA, in R

Then, $AD \parallel CR$,

$\angle\alpha = \angle\alpha'$, [*Corresponding angles*

$\angle\beta = \angle\beta'$ [*Alternate angles*

But $\angle\alpha = \angle\beta$, [Hyp

$\therefore \angle\alpha' = \angle\beta'$,

$\therefore AR = AC.$ [Prop 12.

Again, in $\triangle BCR$,

$\because AD \parallel CR$,

$$\therefore \frac{BD}{DC} = \frac{BA}{AR},$$ [Prop 45.

$$\therefore \frac{BD}{DC} = \frac{BA}{AC}.$$

EXERCISES

1 *Enunciate and prove the converse of this proposition*

2 What does the second part of the proposition become when $AB = AC$?

3 Draw a triangle with sides 5 cm, 6 cm, and 7 cm, calculate the lengths of the segments of the longest side made by the bisector of the opposite angle

4 In Ex 2 find the segments of the base made by the bisector of the opposite external angle

5 In the triangle ABC find the lengths of the segments of the sides made by the internal bisectors of its angles

6 In the triangle ABC the bisector of the exterior angle at A meets BC in O', prove that

$$BO' = \frac{ac}{c-b}, \quad CO' = \frac{ab}{c-b}$$

7 In the triangle ABC the internal and external bisectors of the angle A meet the side BC in O and O' respectively, prove that

$$OO' = \frac{2abc}{c^2 - b^2}.$$

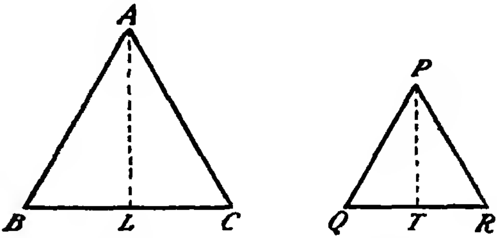
8 The median AD bisects the base of the triangle ABC in D; DB', DC' are drawn bisecting the angles ADB, ADC respectively, and meeting the sides AB, AC in B' and C'; prove that $B'C'$ is parallel to BC

PROPOSITION 50—THEOREM

The ratio of the areas of similar triangles is equal to the ratio of the squares on corresponding sides.

Let ABC, PQR be two similar $\triangle$s having the $\angle$s A, B, $C = \angle$s P, Q, R respectively, then

$$\frac{\triangle ABC}{\triangle PQR} = \frac{BC^2}{QR^2}$$



Let AL, PT be $\perp$ to BC, QR respectively

Then $\triangle$s ABL, PQT are equiangular,

$$\therefore \frac{AL}{PT} = \frac{AB}{PQ} = \frac{BC}{QR} \qquad \text{[Prop 46.}$$

Also $$\frac{\triangle ABC}{\triangle PQR} = \frac{\frac{1}{2}AL}{\frac{1}{2}PT}\frac{BC}{QR} \qquad \text{[Prop 21, Cor 2.}$$

$$= \frac{BC^2}{QR^2}$$

EXERCISES

1 Similar triangles which are equal in area are equal in all respects

2 If the areas of two similar triangles be as 4 : 1, show that the sides of the one are double the corresponding sides of the other

3 With sides 2.4″, 3″, 3.2″, and 3.6″, 4.8″, 4.5″ respectively, draw two triangles and compare their areas

4 If one of two similar triangles has its sides 5 per cent longer than the corresponding sides of the other, compare their areas

5 Two isosceles triangles have equal vertical angles, and their areas have the ratio of 9 : 16, compare their heights

6 A field is represented on a map by a triangle whose area is

1 sq in; if the representative fraction of the scale to which the map is drawn be $\frac{1}{1760}$, find the area of the field in square yards

7 On the sides of any right-angled triangle equilateral triangles are described, prove that the sum of the areas of the triangles described on the sides containing the right angle is equal to the area of the triangle described on the hypotenuse

8 The areas of similar triangles are to one another as the squares of their corresponding altitudes.

9 The areas of similar triangles are to one another as the squares of their corresponding medians

10 If in Ex 7 for "equilateral triangles" we substitute regular pentagons the theorem is true

MISCELLANEOUS QUESTIONS AND EXERCISES —IV.

1 What are *commensurable magnitudes*? Show that the ratio of two commensurable magnitudes can be expressed by means of a fraction, whose numerator and denominator are integers Give an example of two lines whose ratio is not commensurable

2 How many conditions are necessary to define similar rectilineal figures of more than three sides?

3 Under what circumstances are two triangles similar?

4. Draw a triangle ABC, in which the sides opposite the angles A, B, C are 4 8′, 3 6″, and 2 4″ respectively, from a point D in AB distant 1 5″ from A draw the line DE parallel to BC, meeting AC in E Find the lengths of DE and EC

5 In the triangle of Ex 4 the parallel to AB drawn through E meets BC in F, find EF and FC

6 In the triangle of Ex 4 a line is placed between AB and AC parallel to BC, whose length is 3 6″ At what distance from A does it cut the side AB?

7 In the triangle of Exs 4 and 5 DR is drawn parallel to AC, meeting BC in R and cutting EF in O, find OR

8 The sides BC, CA, AB of a triangle are bisected in D, E, F respectively; AD, FE intersect in O, prove that AD and FE bisect one another at O, and that $OF = \frac{1}{2} DC$

9 *In the figure of Ex 8, if the medians CF and AD intersect in G, prove that G is a point of trisection of both medians, and hence prove that the three medians of a triangle pass through the same point.*

10 In the figure of the last exercise prove that the triangles GBC, GCA, and GAB are all equal, and each is equal to one-third of the triangle ABC

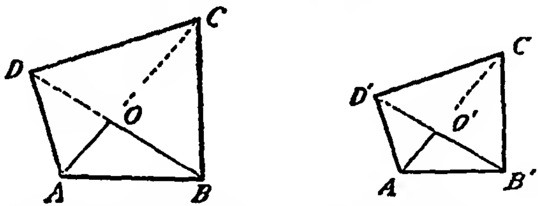
11 ABC, DBC are two equal triangles on the same side of the base BC, show that the sides of these triangles will cut off equal intercepts on any parallel drawn to BC

12 The sides of one triangle are respectively perpendicular to the sides of another, each to each, prove that the triangles are similar

13 CA, CB are equal sides of an isosceles triangle, if the right-bisector of CA meet AB produced in D, prove that AC is a mean proportional between AB and AD

14 In the quadrilateral $ABCD$, if the bisectors of the angles A, C intersect in the diagonal BD, then will the bisectors of the angles B, D intersect in the diagonal AC

15 *Consider two similar rectilineal figures $ABCD$, $A'B'C'D'$ in which the angles, taken in the order in which they are written, are equal and the sides about the equal angles proportional*



Take any point O within $ABCD$ and join OA, OB On $A'B'$ describe the triangle $O'A'B'$ equiangular with the triangle OAB

Prove that if lines be drawn from O and O' to the remaining vertices of the two figures, the two figures will be divided into the same number of corresponding similar triangles

16 *The ratio of the areas of two similar rectilineal figures is equal to the ratio of the squares on corresponding sides*

17 Two quadrilaterals are similar, and their corresponding sides are in the ratio of 4 5, compare their areas

18 The radius of one circle is double that of another, compare the areas of regular hexagons inscribed in them

19 Compare the areas of regular hexagons described in and about a given circle

20 In and about a given circle squares are described, compare their areas

21. On the sides of a right-angled triangle three regular hexagons are described, prove that the sum of the areas of the hexagons on the sides containing the right angle is equal to the area of the hexagon on the hypotenuse

22 Given two regular hexagons, construct another hexagon whose area is equal to the sum of the areas of the given hexagons

23 On bases 5 2 cm and 4.8 cm draw two equilateral triangles, and construct a third equilateral triangle whose area is equal to the difference of the areas of the two triangles.

24. On the sides of a right-angled triangle regular polygons of the same number of sides are described, prove that the sum of the areas of the polygons on the sides containing the right angle is equal to the area of the polygon on the hypotenuse.

25 In the triangle ABC, $AB=2AC$; the bisector of the angle A meets the opposite side in D, DE and DF are drawn parallel to the sides meeting AC and AB in E and F, prove that $AEDF$ is a rhombus.

26 Two triangles which have their sides parallel, each to each, are similar.

27 The perimeters of similar polygons are proportional to their corresponding sides

28 *If any point O be joined to the vertices of a triangle ABC, and if on the straight lines OA, OB, OC points A′, B′, C be taken, so that*

$$\frac{OA'}{OA}=\frac{OB'}{OB}=\frac{OC'}{OC}=n,$$

n being any given number, then the triangle A′B′C′ will be similar to the triangle ABC.

29. Prove that the theorem of Ex 28 is true when the points A', B', C' are situated on the prolongations of the lines AO, BO, CO through the point O.

30 *Enunciate and prove a theorem, similar to the theorem of Ex 28, for any polygon.*

BOOK II

CONSTRUCTIONS

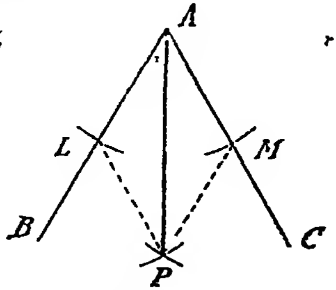
DEF —*A proposition in which a specified geometrical construction is required to be effected is called a* **Problem**

In a problem certain parts of a geometrical figure are given, and the remaining parts are required to be drawn

DEF —*The essential parts of a problem are the* **data**, *or things given, and the* **quaesita**, *or things required*

The student ought to bear in mind that in Demonstrative Geometry all constructions are required to be effected with the help of the **ruler** and **compasses** only.

PROPOSITION 51.—PROBLEM



To bisect a given angle

Let BAC be the given angle

With centre A, and with any radius, describe an arc cutting AB, AC in L, M.

With centres L and M, and with any equal radii, describe arcs cutting in P.

Then the line AP bisects the angle A.

Join LP, MP

In $\triangle$s ALP, AMP,

$$\begin{cases} AL = AM, & \text{[Const.]} \\ LP = MP, & \text{[Const.]} \\ AP = AP, \end{cases}$$

$\therefore$ $\triangle$s are congruent, [Prop 13

and $\angle LAP = \angle MAP$.

EXERCISES

1 *With the help of this proposition an angle may be divided into 2^n equal parts, where n is any integer*

2 Divide an angle into two parts which are in the ratio of 3 : 1

3 Draw an angle of 150° and divide it into eight equal parts

4 Draw an angle of 160°, divide it into four equal parts, and check your construction by measurement

5 In the figure of this proposition AP bisects the angle LPM

6 The figure $ALPM$ is a kite in which the diagonal AP bisects the diagonal LM at right angles

7 Show that AP is the right-bisector of all parallels to LM, which are terminated by the sides of the kite $ALPM$, and hence AP is the axis of symmetry of the figure

8 Show that the bisectors of the angles ALM, AML meet on AP

9 If from any point on the bisector of an angle perpendiculars be drawn to its arms, these perpendiculars are equal

10 With sides 2 9″, 3 2″, and 3 5″, draw a triangle, and draw the bisectors of its internal angles

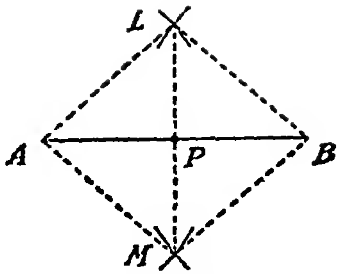
11 In the triangle of the last exercise draw the bisectors of the three external angles, and prove that they are at right angles respectively to the bisectors of the corresponding internal angles

PROPOSITION 52 —PROBLEM

To bisect a given finite straight line

Let AB be the given line

With A and B as centres, and with the same radius (greater than half AB), describe arcs cutting in L and M



Join LM, cutting AB in P

Then AB is bisected in P

Join AL, AM, BL, BM

$\because$ $AL = BL$, [*Const.*

$\therefore$ the point L lies on the right-bisector of AB. [*Prop* 30

Similarly, the point M lies on the right-bisector of AB.

Therefore the line LM is the right-bisector of AB.

Hence AB is bisected in P

COR.—*Bisect a given arc of a circle*

EXERCISES

1 *Show that by repeating this construction we can divide a given line into 2^n parts, where n is any integer*

2 Prove that AB is the right-bisector of LM

3 The figure $ALBM$ is a rhombus, of which AB and LM are axes of symmetry

4 Prove that there is only one point in which a given straight line is bisected

5 The common chord of two intersecting circles is bisected at right angles by the line of centres

6 Draw the triangle ABC, in which $b = 5$ cm, $c = 5\ 3$ cm, and $\angle A = 60°$; construct the right bisectors of the sides of the triangle, and prove that they meet in a point

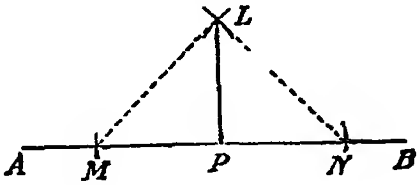
PROPOSITION 53—PROBLEM

To draw a straight line at right angles to a given straight line from a given point in it

Let AB be the given line, and let P be the given point in it.

Cut off $PM = PN$

With M and N as centres, and with the same radius (greater than MP), describe arcs cutting in L.



Join PL

Then $PL \perp AB$

$ML = NL$, [*Const*

. the point L lies on the right-bisector of MN [*Prop* 30

Also, $PM = PN$, [*Const.*

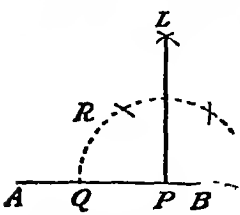
the point P lies on the right-bisector of MN.

Therefore PL is the right-bisector of MN.

Hence $PL \perp AB$

EXERCISES

1 When the point P, from which the perpendicular is to be drawn, lies near one end B of the given line, make the following construction —



With centre P, and any radius, describe a circle cutting AB in Q, step off QR, equal to the radius Produce QR to L, making RL=QR Join PL

Prove that PL is at right angles to AB

2 Take AB 4″ long, and a point P in it distant 2″ from B, make the construction of the last exercise, taking $PQ = RL = 1″$

3 Construct a right-angled isosceles triangle in which each of the equal sides is 2 3″ long

4 Construct a right-angled triangle whose base is 1 5″ and hypotenuse 3″, show that one of the acute angles is double of the other.

5 In a given straight line find a point equidistant from two given points

When is the solution impossible?

Describe a circle which shall pass through two given points and have its centre on a given straight line

6 Construct an angle of 45°

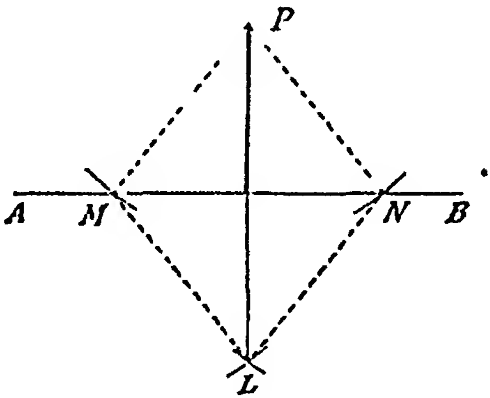
7 Find a point equidistant from the angular points of an isosceles trapezium

Note —*A trapezium whose non-parallel sides are equal is called* an isosceles trapezium

Proposition 54 —Problem

To draw a straight line perpendicular to a given straight line of unlimited length from a given point outside it

Let AB be the given line, and let P be the given point



With P as centre, describe a circle cutting AB in M and N.

With M and N as centres, and any equal radii, describe arcs cutting in L.

Join PL.

Then $$PL \perp AB$$

$\because PM = PN,$

$\therefore P$ lies on the right-bisector of MN [*Prop* 30.

Again, $ML = NL,$ [*Const.*

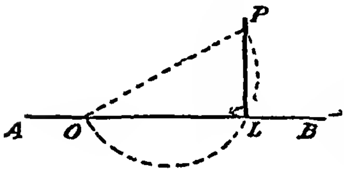
L lies on the right-bisector of MN [*Prop* 30

Therefore PL is the right-bisector of MN

Hence $PL \perp AB$

EXERCISES

1 *Take any point O in AB, on OP as diameter describe a semicircle cutting AB in L, prove that PL is perpendicular to AB*



2 Draw a triangle with sides 2 1", 2 4", and 2 6", describe semicircles on the sides situated internally, and hence draw the three altitudes of the triangle

3 Repeat the last exercise with a triangle of sides 14 cm, 15 cm, and 16 cm

4 *Given two points on opposite sides of a given straight line Through the given points draw two straight lines which shall meet in the given line and contain an angle bisected by that line*

5 *Given two points on the same side of a given straight line Through the points draw two straight lines which shall meet in the given line and make equal angles with it*

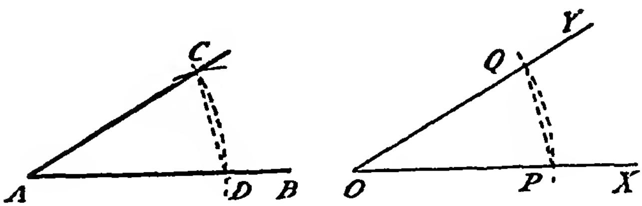
PROPOSITION 55 —PROBLEM

At a given point in a given straight line to make an angle equal to a given angle

Let AB be a given line, and A a point in it at which it is required to make an angle equal to the given angle XOY

With O as centre, describe an arc cutting OX, OY in P and Q

With A as centre, and radius equal to OP, describe an arc cutting AB in D, and with D as centre, and radius



equal to PQ, describe an arc cutting the arc whose centre is A in C

Join AC.

Then $\angle BAC = \angle XOY$.

In the $\triangle$s DAC, POQ,

$$\because \begin{cases} AD = OP, \\ AC = OQ, \\ DC = PQ, \end{cases}$$ [Const.

$\therefore$ $\triangle$s are congruent, [Prop 13.

and $\angle DAC = \angle POQ$

Therefore $\angle BAC = \angle XOY$

EXERCISES

1 Make an angle double of a given angle ✓

2 At a given point in a given straight line make an angle equal to the supplement of a given angle

3 At a given point in a given straight line make an angle equal to the complement of a given angle

4. Draw an isosceles triangle, having given the vertical angle and one of the equal sides.

5 Construct an isosceles triangle having a base 2″ long and each of the angles at the base of 45°

6 Construct a right-angled triangle, having given the hypotenuse and one of the acute angles ✓

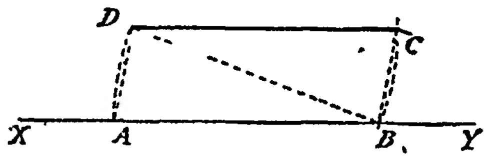
7 Construct a right angled triangle, having given the base and the acute angle at the base.

8 *Show that a right angled triangle can be divided into two isosceles triangles*

PROPOSITION 56 —PROBLEM

To draw a straight line, through a given point, parallel to a given straight line

Let it be required to draw through the point D a straight line parallel to XY



With D as centre, and any radius, draw an arc BC cutting XY in B With B as centre, and the same radius as before, draw an arc DA cutting XY in A With centre B and radius equal to AD, draw an arc cutting the arc BC in C Join DC

Then $DC \parallel XY$

Join DB, AD, BC

Then in $\triangle$s ABD, CDB,

$$\begin{cases} AB = CD, & \text{[Const]} \\ AD = BC, & \text{[Const]} \\ BD = BD, & \end{cases}$$

$\triangle$s are congruent, [*Prop* 13

and $\angle ABD = \angle BDC$,

but these are alternate angles,

$\therefore$ $DC \parallel XY$ [*Prop* 4

EXERCISES

1 Through a given point draw a line making with a given line an angle equal to a given angle.

Show that two such lines can be drawn

2 Through a given point between two intersecting straight lines draw a straight line that shall be bisected at the point

3 Through a given point between two intersecting straight lines draw a straight line which shall form with the given lines an isosceles triangle.

4 Through a given point draw a straight line, such that the intercept made on it by two given parallel lines shall be equal to a given line

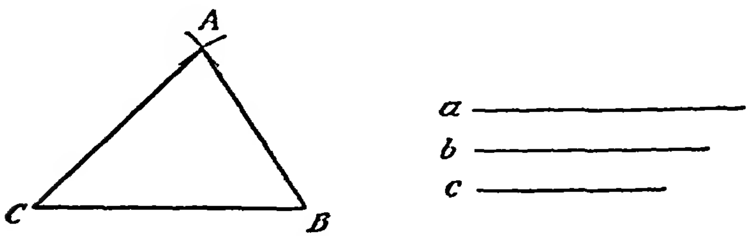
5 Through two given points in two parallel lines draw straight lines which shall form a rhombus with the parallels

6 Between two given intersecting lines place a line which shall be parallel to one given line and equal to another

PROPOSITION 57 —PROBLEM

To construct a triangle having its sides equal to three given straight lines, any two of which are together greater than the third

Let it be required to draw a triangle whose sides are equal to the three lines a, b, c



Take the line BC equal in length to a

With centres B and C, and radii equal in length to c and b respectively, describe arcs of circles intersecting in A

Then ABC is the required triangle.

EXERCISES

1 With sides 5 cm , 7 cm , and 9 cm describe a triangle, and draw its medians

2 With sides 2 5″, 2 8″, and 3 3″ draw a triangle, and construct its three altitudes.

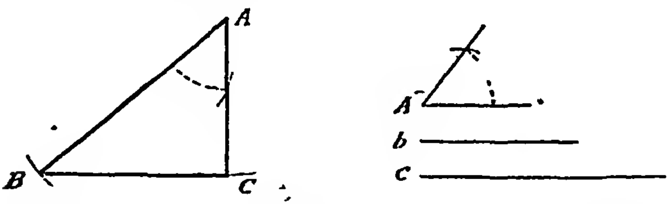
3 Draw a triangle with sides 4 6 cm , 5 cm , and 5 4 cm , and draw also the internal bisectors of its angles

4 Draw a triangle having its sides 3 1″, 4″, and 3 7″ long, and construct the right-bisectors of its three sides

Proposition 58 —Problem

To construct a triangle, given two sides and the included angle.

Let it be required to construct a triangle, two of whose sides are equal in length to b and c, and the angle contained by these sides equal to the angle A



Make $\angle BAC = \angle A$ [*Prop* 55.

Cut off $AB = c$ and $AC = b$.

Join BC

Then ABC is the required triangle

EXERCISES

1 Construct the triangle ABC, having given $BC = 3''$, $CA = 2.6''$, and $\angle C = 45°$

2 Construct the triangle ABC, having given $c = 7$ cm , $a = 5\ 6$ cm , and $\angle B = 30°$

3 Draw an isosceles triangle having each of its equal sides 4″ long and its vertical angle of 120°

Find the length of the perpendicular from the opposite angle on the longest side

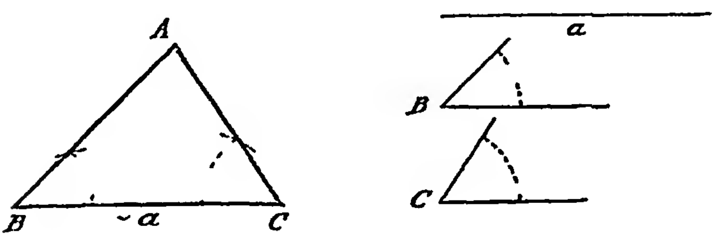
4. Construct a rhombus of 2 5″ side, and having an angle of 60°

Prove that one of its diagonals is 2 5″ long

PROPOSITION 59 —PROBLEM

To construct a triangle, given a side and two angles

Let it be required to construct a triangle *ABC*, of which the parts *a*, *B*, and *C* are given



Take *BC* equal to *a*

Make the angles *ABC*, *ACB* equal to the given angles *B* and *C* respectively. [*Prop.* 55.

Then *ABC* is the required triangle

EXERCISES

1. Construct the triangle *ABC*, having given $b=6$ cm, $\angle C=30^\circ$, $\angle A=60^\circ$.

2 Construct the triangle *ABC*, having given $c=4''$, $\angle C=45^\circ$, $\angle A=60^\circ$

Draw a triangle similar to it and containing four times its area.

3 Draw the triangle *ABC*, in which $a=3\ 6$ cm, $B=30^\circ$, $A=135^\circ$

Construct a triangle similar to it and containing one fourth its area

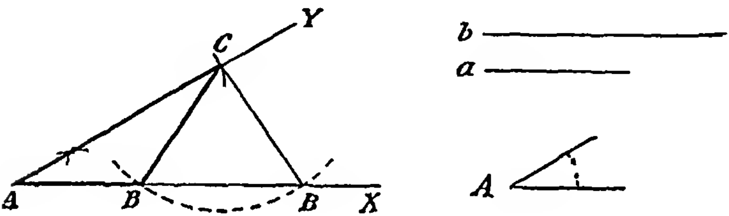
4. On a base 6 6 cm long construct an isosceles triangle whose vertical angle is of 30°.

Proposition 60.—Problem

To construct a triangle, given two sides and the angle opposite one of them

Let it be required to construct a triangle ABC, of which the parts a, b, and A are given

Make the angle XAY equal to A.



Cut off $AC=b$ With centre C, and radius equal to a, describe a circle cutting AX in B

Then ABC is the required triangle

EXERCISES

In the figure of this proposition draw CL perpendicular to AB, and let $CL=p$ Show that—

(i) If $a<p$ the triangle cannot be drawn

(ii) If $a=p$ one triangle can be drawn with the given parts, and that triangle is right-angled

(iii) If $a>p$ there are three cases to be considered —

1 When $a<b$ two triangles ABC, $AB'C'$ can be drawn, as in the figure

2 When $a=b$ one triangle can be drawn, for the point B will coincide with A

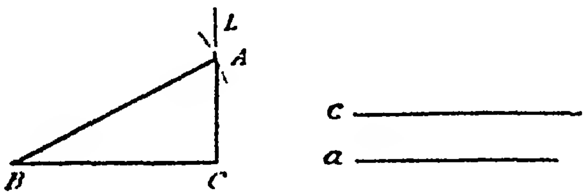
3 When $a>b$ one triangle can be drawn

Make drawings to illustrate all these cases

Proposition 61.—Problem

To construct a right-angled triangle, given the hypotenuse and one side

Let c be the given hypotenuse and a the given side



Let $BC = a$ Draw CL perpendicular to BC With centre B and radius equal to c describe a circle cutting CL in A

Then ABC is the required triangle.

EXERCISES

1. Construct a right-angled triangle whose hypotenuse is 3″ and base 2.4″, on the hypotenuse describe a semicircle lying inwards

2 The hypotenuse of a right angled triangle is 3″ long, and one of the acute angles is of 30°, construct it, and prove that the shortest side is 1 5″ long

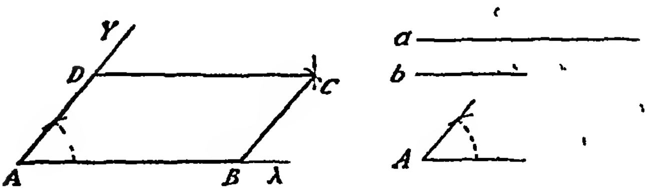
3 With sides 3 4″, 1 6″, and 3″ construct a triangle, and prove that it has a right angle.

4 Construct a right angled isosceles triangle whose hypotenuse is 4″ long, prove that the perpendicular from the right angle on the hypotenuse is 2″ in length

Proposition 62.—Problem

To construct a parallelogram, given two sides and the included angle

Let a and b be the two given sides and A the given angle



Make the $\angle XAY = \angle A$ Cut off AB and AD equal to a and b respectively

Through B and D draw BC, DC parallel to AD and AB respectively

Then $ABCD$ is the required parallelogram

Cor —*On a given straight line construct a square*

EXERCISES

1 With sides 5 cm and 3 5 cm construct a parallelogram which has an angle of 75°

2 Construct a parallelogram of which the diagonals are 8 cm and 12 cm, and one of the sides is 7 cm

3 Draw a parallelogram of which the adjacent sides are 3 5" and 4", and the diagonal which meets them is 5 6"

4 Construct a parallelogram whose diagonals are 8 cm and 10 cm, and intersect one another at an angle of 45°

Proposition 63 —Problem

To divide a given straight line into a given number of equal parts or into parts in any given proportion

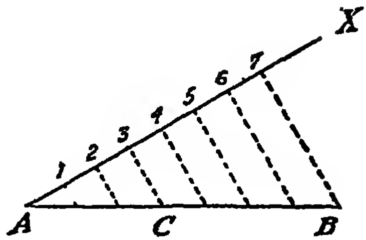
(1) Let it be required to divide the line AB into seven equal parts

Draw AX, making any angle with AB On AX step off seven equal distances A–1, 1–2, 6–7

Join $B7$

Through 1, 2, 3, 4, 5, 6 draw parallels to $B7$.

Then these parallels will divide AB into seven equal parts [*Prop* 19



(ii) Let it be required to divide AB in the ratio of 3 4

Make the same construction as before, then the parallel $3C$ will divide AB in C in the ratio 3 : 4

For AB is divided into seven equal parts, of which AC contains 3 and CB contains 4.

Therefore AC $CB = 3$ 4.

EXERCISES

1 Divide a line 7 2 cm long into eight equal parts, and check by measurement

2 Divide a line 3 5″ long into parts in the ratio of 2 5, and check by measurement

3 Divide a line 5 4″ long into three parts in the ratio of 1 3 5; check by measurement.

PROPOSITION 64—PROBLEM

To construct a triangle equal in area to a given polygon

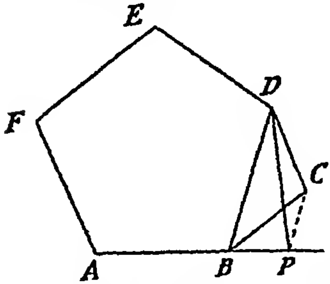
Let it be required to construct a triangle equal in area to the polygon $ABCDEF$

Join BD

Draw $CP \parallel BD$, meeting AB produced in P

Join PD.

Then *APDEF* is a five-sided figure equal in area to the six-sided given figure *ABCDEF*



For the Δs *BDP* and *BDC* are on the same base *BD* and are between the same parallels,

$\therefore \Delta BPD = \Delta BDC.$ [*Prop* 21, *Cor.*

To each of these equals add the figure *ABDEF*, then fig *APDEF* = fig *ABCDEF*

Thus we have constructed a figure equal in area to the given figure, but having one side less

By repeating this construction the given figure will finally be reduced to a triangle.

EXERCISES

1 Draw a square on a base 2″ long, and reduce it to a right-angled triangle of equivalent area

2 In a circle of 4 cm radius inscribe a regular hexagon, and reduce it to an isosceles trapezium of equivalent area.

PROPOSITION 65—PROBLEM

To construct a parallelogram equal in area to a given triangle, and having one of its angles equal to a given angle

Let *ABC* be the given Δ, and let the angle of the required $\|^m$ be equal to the $\angle \alpha$.

Bisect *BC* in *M*. Make $\angle CMD = \alpha$.

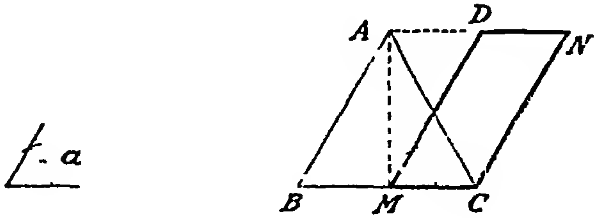
Draw *CN* $\|$ *MD* and *ADN* $\|$ *BC*

Then *MCND* is the required $\|^m$

Join *AM*.

For $\because$ $\|^{m}$ MN and $\triangle ABC$ are on the same base and between the same parallels,

$$\therefore \|^{m} MN = 2\triangle AMC \qquad [Prop. 20, Cor.$$



Also $\because$ $\triangle$s AMC, AMB are on equal bases and of the same altitude,

$$\triangle AMC = \triangle AMB, \qquad [Prop. 21, Cor.$$

$$\therefore \triangle ABC = 2\triangle AMC,$$

$$\therefore \|^{m} MN = \triangle ABC,$$

and its $\angle DMC = a$.

EXERCISES

1 Draw a triangle with sides 1 7", 2 4", and 1 9"; and on a base 1 2" construct a parallelogram of equivalent area, having one of its angles equal to that of an equilateral triangle

2 With sides 5 cm, 7 cm and 8 5 cm draw a triangle; construct a rhombus of equivalent area, having one of its diagonals 3 5 cm long

MISCELLANEOUS EXERCISES.—V.

1 Draw an angle of 84° with the help of the protractor, bisect it by construction and check by measurement

2 Draw an angle of 54°, using the protractor, and construct an angle double of it

3 Draw a line 54 mm long and bisect it, check your construction by measurement

4 Draw an arc of a circle subtending an angle of 150° at the centre; bisect it.

5 Draw an arc of a circle subtending an angle of 120° at the centre of the circle, bisect it by construction, and check the correctness of your drawing by measuring angles at the centre

6 On a base 5 cm long construct a square

7 Draw a triangle with sides 11 cm, 13 cm, and 15 cm, construct its longest altitude.

8 Draw an angle of 37°, and find another equal to it by construction

9 Given a straight line X, and a point P distant 2" from it, through P draw two straight lines making angles of 30° with X

10 In the figure of the last exercise draw two lines through P making angles of 120° with X

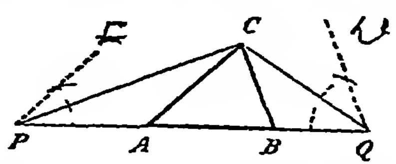
11 Given a straight line and a point lying without it, through the point draw two straight lines, making a right-angled isosceles triangle with the given line

12 Construct a parallelogram when the two diagonals and a side are given

13 Construct a rectangle when a diagonal and a side are given.

Analysis—Very often the solution of a problem is most easily effected by assuming the problem solved, and then trying to find out some relation between the *quaesita* and the *data* which, with the help of known theorems and constructions, will enable you to draw the figure This is known as the *method of analysis*, and it will be best understood by following a few examples

14 *Construct a triangle, having given the perimeter and the angles*
Suppose the triangle ABC to be drawn as required by the problem.



We first try to find a line in the figure equal to the given perimeter. With this end in view produce AB both ways, and cut off $BQ=BC$ and $AP=AC$

Then PQ is equal to the perimeter

Now examine the figure a little further

$\because$ BCQ is an isosceles Δ,

$\angle BQC = \angle BCQ$,

$\therefore$ $\angle CBA = 2\angle BQC$ [*Prop* 7, *Cor*

Similarly, $\angle CAB = 2\angle CPA$.

Thus the angles at P and Q are each halves of given angles

Therefore in the ΔCPQ the base is given, and the angles at the base are also given, hence the triangle can be constructed

From the foregoing analysis we arrive at the following construction :—

Construct a triangle CPQ, in which PQ = given perimeter, $\angle$s P, $Q = \frac{1}{2}$ of given angles B, C. [*Prop* 59.

Make the angles PCA, QCB = angles at P, Q respectively

Then ABC is the required triangle.

The student can easily supply the proof

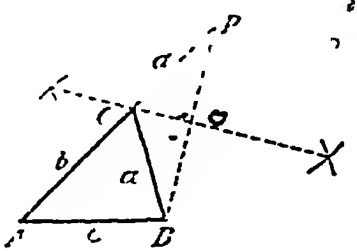
15. The perimeter of a triangle is 8", and the angles at its base are of 45° and 60°; construct the triangle

16 With sides 3", 3 4", and 3 9" draw a triangle, and construct another triangle equiangular with it, and having a perimeter of 6"

17 Construct an isosceles triangle whose perimeter is 7 5" and vertical angle of 30°.

18 The perimeter of a right-angled triangle is 12 cm, and one of its acute angles is of 60°, construct it

19 *Given the base of a triangle, the sum of the other two sides, and one of the angles at the base, construct the triangle*



Suppose ABC is the required triangle, in which $a+b$, c, and A are given. We first construct the line $a+b$ in the figure. Produce AC, cutting off $CP = a$

Then $AP = a + b$

Now, in ΔAPB, AP, AB, and the included angle are given, hence the triangle can be constructed

Again notice that since CPB is an isosceles triangle the vertex C must lie on the right-bisector of its base BP

From this analysis we obtain the following construction —

With sides c, $a+b$, and included angle A draw the triangle APB [*Prop* 58.

Let the right bisector of BP meet AP in C.

Join BC

Then ABC is the required triangle.

Let the student complete the proof

20 Given the base of a triangle, the difference of the other two sides, and the lesser angle at the base, construct the triangle.

[Here $b-a$, c, and A are given]

21 The base of a triangle is $5''$, the sum of the other two sides is $7''$, and one of the angles at the base is of $60°$, construct the triangle

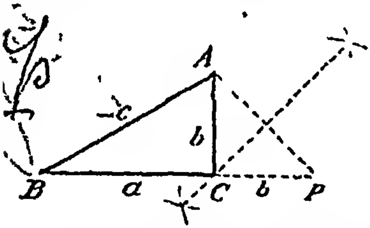
22 The base of a triangle is $4.8''$, the difference of the other two sides is $1''$, and one of the angles at the base is of $75°$, construct the triangle

23 Describe an isosceles triangle of given base and perimeter

24 Construct a triangle, having given a side, the angle opposite to the side, and the sum of the other two sides

25 Construct a triangle, having given a side, the angle opposite to the side, and the difference of the other two sides

26 *Construct a right angled triangle, having given the hypotenuse and the sum of the other two sides*



Let ABC be the required triangle, having the angle at C a right angle

Here c and $a+b$ are given, we therefore proceed to lay down the line $a+b$ in the figure as shown.

Now notice that in the triangle ABP, AB, BP are given, and the angle at P is also given, since it is half a right angle, hence the triangle can be constructed [Prop 60

Again notice that the point C lies on the right bisector of AP

The student can easily complete the solution

27 Construct a right angled triangle, having given the hypotenuse and the difference of the other two sides

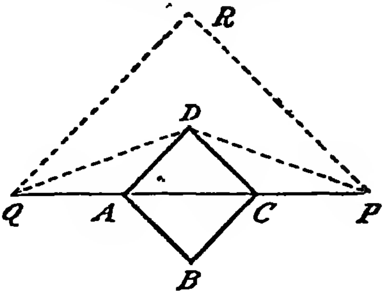
28 *Construct a triangle, having given the altitude and the angles at the base*

Analysis —Through the vertex of a triangle draw a parallel to the base This parallel makes with the sides angles which are equal to the angles at the base of the triangle

29 The altitude of a triangle is $3''$, and the angles at its base are of $45°$ and $60°$, construct it

30 Construct a square, having given the difference between the diagonal and a side.

31 *Construct a square, having given the sum of the diagonal and two sides*



Let $ABCD$ be the square constructed as required

Produce AC both ways and cut off $AQ=CP=$a side

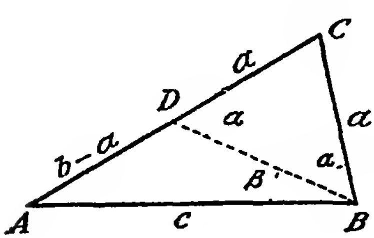
Then $PQ=$sum of the diagonal and two sides

Draw QR, $PR \parallel AD$, CD, meeting in R

Now notice that since AQD is isosceles, QD bisects the angle at Q Hence the following construction —

On PQ as hypotenuse describe an isosceles right-angled triangle QRP Let the bisectors of the angles at P and Q meet in D Draw DA, DC parallel to RQ, RP, then DA, DC are two sides of the square.

32 *Construct a triangle, having given the base, the difference of the other two sides, and the difference of the angles opposite to those sides*



Suppose ABC is the Δ constructed as required

Here c, $b-a$, and $B-A$ are given

We proceed to find on the figure $b-a$ and $B-A$

From CA cut off $CD=CB$,

then $AD=b-a$

We have

$$\alpha+\beta=B \text{ and } \alpha-\beta=A,$$

[*Prop* 7, *Cor.*

$$\therefore \alpha=\tfrac{1}{2}(B+A) \text{ and } \beta=\tfrac{1}{2}(B-A)$$

These two results are important

Now in ΔABD two sides and the angle opposite to one of them is given, hence the triangle can be constructed [*Prop* 60

33 Construct a triangle whose base is 4″, the difference of the other two sides is 2 5″, and the difference of the angles opposite to those sides is 30°

34. Construct a triangle, having given the base, the sum of the other two sides, and the difference of the angles opposite to those sides

35 Construct a triangle whose base is 1 8″, the sum of the other two sides is 2 9″, and the difference of the angles opposite to those sides is 30°.

36 Construct a triangle, having given one side, the angle opposite to that side, and the difference of the other two angles

37 Construct the triangle ABC, in which $a=2\ 8''$, $A=75°$, and $B-C=15°$

38 In a right angled triangle inscribe a square, having one of its angles coinciding with the right angle

39 In the last exercise prove that if s be the side of the inscribed square, a, b the sides containing the right angle, and Δ the area of the triangle, then

$$s(a+b)=2\Delta$$

40 In a given triangle inscribe a rhombus, having one of its angles coinciding with an angle of the triangle

41 In a given triangle inscribe a square

42 In the last exercise, if s be the side of the inscribed square and a and p the base and altitude of the triangle, prove that

$$\frac{1}{s}=\frac{1}{a}+\frac{1}{p}$$

43 Construct a triangle, having given the base, the altitude, and one of the angles at the base

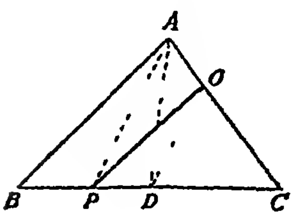
44 Construct a triangle, having given the base, the altitude, and one of the remaining sides

45 Construct a triangle of height $2''$ on a base $3''$ long, having a side $2\ 5''$, inscribe a square in the triangle, having a side coincident with the base of the triangle.

Find the length of a side of the square

46 *Bisect a triangle by a straight line drawn through a given point in one of its sides*

Let P be the given point in the side BC



Suppose that the line PO divides the given triangle into two equal parts, so that

$$\Delta OPC=\tfrac{1}{2}\Delta ABC$$

Draw the median AD, then

$$\Delta ADC=\tfrac{1}{2}\Delta ABC,$$
$$\Delta OPC=\Delta ADC$$

From each of these equals take away ΔODC, then

$$\Delta OPD=\Delta OAD$$

Hence $$OD \parallel AP.$$

From the above analysis we derive the following construction —

Bisect BC in D and draw DO parallel to AP, meeting CA in O. Then PO is the required line.

The student can supply the proof of this construction

47 With sides 3″, 3 5″, and 2 8″ draw a triangle, from a point in the first side, distant 1″ from the largest angle, draw a line to bisect the triangle.

48 Bisect a quadrilateral by a straight line drawn through one of its angular points

49 Take a line BD 4″ long, on opposite sides of BD describe the triangles BAD, BCD such that $AB=1\ 9''$, $AD=3''$, $BC=2\ 4''$, and $CD=3\ 5''$

Through A draw a straight line to bisect the quadrilateral $ABCD$

50 Construct the rectilineal figure $ABCDE$, in which $AB=3''$, $BC=2''$, $CD=1\ 5''$, $DE=2\ 2''$, $EF=3\ 2''$, $AC=AD=5''$ Make a triangle of equivalent area, having its base coincident with AB

51 Bisect a given triangle by a straight line drawn parallel to the base

52 Bisect a given triangle by a straight line drawn perpendicular to the base

53 Bisect a parallelogram by a straight line drawn through a given point without the parallelogram

54 Two parallelograms being given in magnitude and position, draw a line which will bisect them both

55 Construct a triangle, having given the middle points of the three sides

56 Construct a right-angled triangle, having given the perpendicular from the right angle on the hypotenuse and a side

57 Construct a triangle, being given the angles and the sum of two sides

58 Construct a triangle, being given the angles and the difference of two sides

59 Construct a triangle, being given two sides and the median which passes through their point of intersection

60 In a given square inscribe an equilateral triangle having one of its angular points coincident with an angular point of the square

61 Construct a square, having given the sum of a side and a diagonal

62 Construct a parallelogram, having given a diagonal, a side, and an angle.

63 Describe a right-angled triangle, having given the hypotenuse and the perpendicular drawn to it from the right angle

64 From the vertex of a triangle draw a straight line to meet the base, which shall exceed the lesser side of the triangle as much as it is exceeded by the greater

65 Draw a straight line FE parallel to the base of the triangle ABC, meeting the sides AB, AC in F and E, such that (i) $FE=FB$, (ii) $FE=BF+CE$, (iii) $FE=BF\sim EC$

66 Draw a straight line to cut off a kite from a given triangle

In how many different ways can this be done?

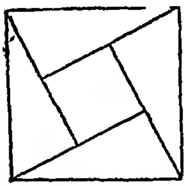
67 Construct a polygon equal in all respects to a given polygon

68 Construct a rhombus equal to a given triangle and having a side equal to a side of the triangle

69 Bisect a quadrilateral by a straight line drawn through a given point in one of its sides

70 One acute angle of a triangle is double of another, divide the triangle into two isosceles triangles

71 Within a triangle find a point, such that straight lines drawn from it to the angular points shall trisect the triangle



72 Divide a given square into five equal parts, consisting of one square and four right-angled triangles

73 Construct a square equal to three-fourths of a given square

74 Between two given straight lines inflect a straight line, which shall be equal to one given line and parallel to another

75 Construct a parallelogram which shall have the same area and perimeter as a given triangle

PROPOSITION 66—PROBLEM

To draw a tangent to a given circle

(i) at a given point on the circumference,

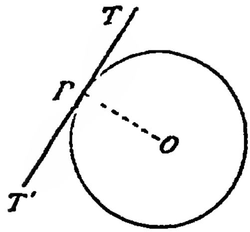
(ii) from a given point without it

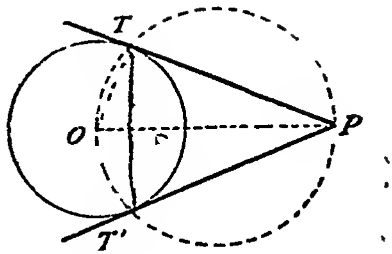
Let it be required to draw through the point P a straight line which will touch the circle whose centre is O.

(i) When P is on the circumference

Join OP, and draw $PT \perp OP$

Then PT is the tangent required. [*Prop* 37, *Cor*





(ii) When P is without the circle

Join OP, and on OP as diameter describe a circle cutting the given circle in T, T'.

Join PT, PT'

Then PT, PT' are the required tangents.

Since OTP is a semicircle, the $\angle OTP$ is a rt. $\angle$

[*Prop* 41.

Therefore TP is a tangent. [*Prop* 37, *Cor.*

Similarly, $T'P$ is a tangent.

EXERCISES

1 *The two tangents drawn to a circle from an external point are equal*

2 *The two tangents drawn to a circle from an external point subtend equal angles at the centre*

3 *The two tangents drawn to a circle from an external point make equal angles with the line joining the external point to the centre*

4. *The chord of contact of tangents drawn from an external point to a circle is bisected at right angles by the line joining the point to the centre*

5 The figure $OTPT'$ is a kite, of which OP is the axis of symmetry

6 From a point distant c from the centre of a circle of radius r, a pair of tangents is drawn to the circle, show that

(i) the length of each tangent is $\sqrt{(c^2-r^2)}$,

(ii) the distance of the chord of contact from the centre is r^2/c;

(iii) the length of the chord of contact is $2r\sqrt{(c^2-r^2)}/c$

7 Any two tangents to a circle are equally inclined to their chord of contact

8 The bisector of an angle is the locus of the centres of circles which touch both the arms of the angle

9 Describe a circle of given radius to touch two given straight lines

10 If the sides of a quadrilateral touch a circle, the sum of the angles which each pair of opposite sides subtends at the centre is equal to two right angles.

11 *The segment of a variable tangent intercepted between any two fixed tangents to a circle subtends a constant angle at the centre of the circle*

12 *The perimeter of the triangle formed by two fixed tangents, and the segment of a variable tangent intercepted between them, is constant*

13 *If a quadrilateral circumscribe a circle, the sum of one pair of opposite sides is equal to the sum of the other pair*

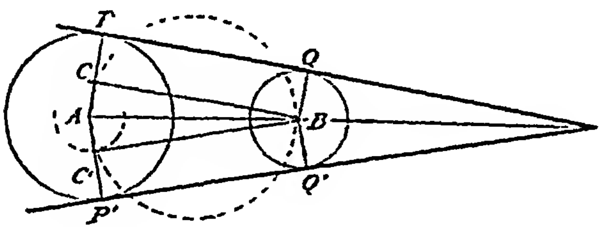
DEF —*A common tangent to two circles whose points of contact are on the same side of the line of centres is called a direct* common tangent.

DEF —*A common tangent to two circles whose points of contact are on opposite sides of the line of centres is called a* transverse common tangent.

PROPOSITION 67 —PROBLEM

To draw the common tangents to two given circles.

(1) Direct common tangents



Let A and B be the centres of the circles whose radii are R and r, A being the centre of the greater circle.

On AB as diameter describe a circle, and with centre A, and radius equal to the difference of the radii of the two circles, describe another circle cutting the first in C, C'.

Draw the radius ACP

Draw the radius $BQ \parallel AP$.

Join PQ.

Then PQ is a direct common tangent to the two circles.

$$\because AC = R - r,$$
$$\therefore CP = R - (R - r)$$
$$= r = BQ.$$

Hence $CBQP$ is a $\parallel^m$

Also, $\because \angle ACB$ in a semicircle is a rt $\angle$,

$\therefore$ all the angles of the $\parallel^m$ $CBQP$ are rt $\angle$s

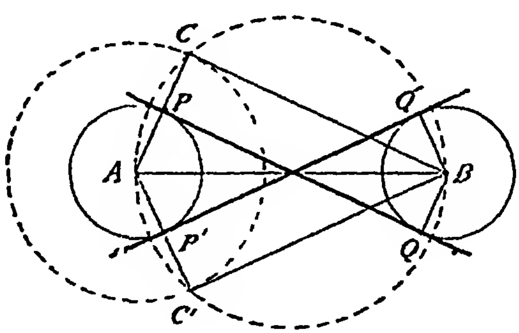
Since the angles APQ, BQP are right angles,

PQ touches both circles

Similarly, it can be proved that $P'Q'$ touches both circles.

(ii) Transverse common tangents

As before, on AB as diameter describe a circle, and



with centre A, and radius equal to the sum of the radii of the two circles, describe another circle cutting the first in C, C'.

Draw the radius APC

Draw the radius $BQ \parallel AP$

Join PQ

Then PQ is a transverse common tangent to the two circles

$$AC = R + r,$$
$$\therefore CP = (R + r) - R$$
$$= r = BQ$$

Hence $CBQP$ is a $\|^m$

Also, $\angle ACB$ in a semicircle is a rt. $\angle$,

all the angles of the $\|^m$ $CBQP$ are rt $\angle$s

Since the angles APQ, BQP are right angles,

PQ touches both circles

Similarly, it can be proved that $P'Q'$ touches both circles

EXERCISES

1 *Prove that in both the figures of this proposition the common tangents make equal angles with the line of centres*

2 *Prove that in both figures the common tangents meet the line of centres in the same point*

3 *Prove that the line of centres in each figure is the axis of symmetry.*

4 Draw the direct and transverse common tangents of two circles.

Explain how the transverse common tangents coalesce into one tangent at the point of contact of the two circles.

5 Prove that if c be the distance between the centres of two circles whose radii are R and r, the lengths of their direct and transverse common tangents are given by the equations

$$T^2 = c^2 - (R - r)^2 \text{ and } T'^2 = c^2 - (R + r)^2$$

6 Prove that the direct common tangent of two circles which touch externally is a mean proportional between their diameters

7 Draw the direct common tangents of two circles whose radii are $\frac{3}{4}''$ and $\frac{1}{2}''$, and centres $2''$ apart

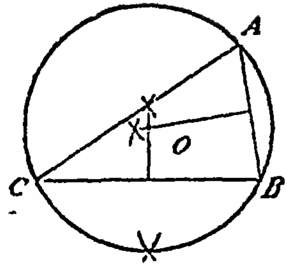
8 Draw the transverse common tangents of two circles whose radii are 2 cm and 1 5 cm., and centres 5 cm apart.

PROPOSITION 68—PROBLEM

To describe a circle about a given triangle

Let ABC be the given triangle.

Let the right-bisectors of AB and BC meet in O, then O is the centre of the required circle



Since O is on the right-bisector of AB,

$$\therefore OA = OB$$

Similarly, $OB = OC$

Hence $OA = OB = OC$.

Therefore a circle whose centre is O and radius OA will pass through A, B, and C

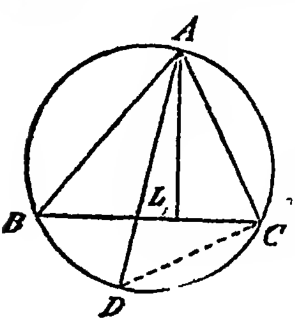
EXERCISES

1. *If from the vertex of a triangle a perpendicular be drawn to the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by the perpendicular and the circumdiameter of the triangle*

In ΔABC let $AL \perp BC$, then

$$AB \cdot AC = AL \cdot AD,$$

where AD is the diameter of the circle ABC



In Δs ABL, ADC,

$\because \begin{cases} \angle B = \angle D, & \text{[Prop 40} \\ \angle ALB = \angle ACD, & \text{[Prop 41} \end{cases}$

$\therefore$ Δs are similar [Prop 46.

Hence $\dfrac{AB}{AL} = \dfrac{AD}{AC}$

Therefore $AB \cdot AC = AL \cdot AD$

2 *Prove that the circumradius of a triangle is given by the formula*

$$R = \frac{abc}{4\Delta}$$

We have proved that

$$bc=2R\ p,$$

where p is the altitude of the triangle

Therefore $$abc=2R\ pa,$$
$$=2R\ 2\Delta$$

Hence $$R=\frac{abc}{4\Delta}$$

3 Find the radius of the circle circumscribing the triangle whose sides are 5", 8", and 5"

4 Draw a triangle with sides 2", 2 1", and 2.9", and describe a circle about it Find the radius of this circle

5 With sides 2 5", 2", and 1 5" draw a triangle, find the position of its circumcentre and the length of its circumradius

6 Prove that the circumradii of similar triangles are in the ratio of their corresponding sides

7 Prove that the radius of the circle which passes through the middle points of the sides of a triangle is equal to half the radius of the circle which passes through the angular points

8 *If O be the circumcentre of the triangle ABC, prove that*

$$\angle BOC=2\angle A$$

9 *The base and vertical angle of a triangle being given, the circumradius is constant*

10 The square described on the side of an equilateral triangle is three times the square on its circumradius

11. Two triangles have one side of the one equal to one side of the other, and have the angles opposite to these sides supplementary, prove that they have equal circumradii

Proposition 69 —Problem

To inscribe a circle in a given triangle.

Let ABC be the given triangle

Draw BI, CI bisecting the angles B and C.

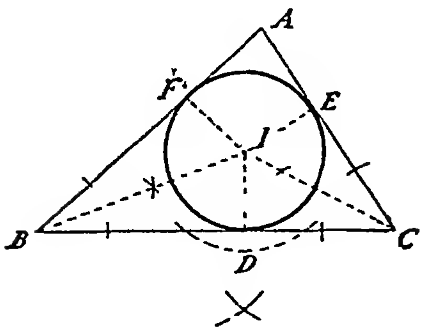
Draw $ID \perp BC$

Then I is the centre and ID the radius of the required circle

Draw IE, $IF \perp CA$, AB respectively.

$\therefore$ BI is the bisector of the $\angle B$,

$ID = IF$. [*Prop* 31.



For a similar reason

$$ID = IE$$

Therefore $ID = IE = IF$

Also the $\angle$s at D, E, F are rt $\angle$s [*Const*

Hence the circle with centre I and radius ID will touch BC, CA, AB. [*Prop* 37.

EXERCISES

1 Prove that AI bisects the angle A.

2 *Prove that*

$$AF+BD+CE=AE+BF+CD=s, \text{ where } 2s=a+b+c$$

3 *Prove that*

$$AF=AE=s-a,\ BF=BD=s-b,\ CE=CD=s-c$$

4 *Prove that the inradius of the triangle ABC is given by*

$$r=\frac{\Delta}{s}$$

5 Prove that the inradii of similar triangles are in the ratio of their corresponding sides

6 The incentre of an equilateral triangle coincides with its circumcentre

7 Any circle whose centre is I, and radius greater than the inradius, will cut off equal chords from the sides of the triangle

8 With sides 3 9", 4 2", and 4 5" draw a triangle, inscribe a circle in the triangle, and calculate the length of its radius

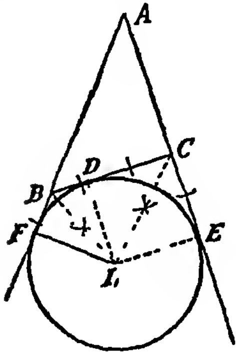
9 In Ex 8 find the lengths of the tangents drawn to the incircle from the smallest angle

10 Draw a triangle with sides 2 6", 2 9", 3 1", inscribe a circle in the triangle, and find the segments of the sides made by the points of contact of the inscribed circle

Proposition 70 —Problem

To escribe a circle to a given triangle

Let ABC be the given triangle We are required to draw a circle to touch the side opposite to the angle A and the other two sides produced



Draw BI_1, CI_1, bisecting the external angles at B and C

Draw $I_1D \perp BC$

Then I_1 is the centre and I_1D the radius of the circle which touches BC and the sides AB and AC produced.

Draw I_1E, $I_1F \perp AC$, AB produced

$\because BI_1$ is the bisector of the ext. $\angle B$,

$I_1D = I_1F$ [*Prop.* 31.

For a similar reason

$$I_1D = I_1E$$

Therefore $I_1D = I_1E = I_1F$.

Also $\angle$s at D, E, F are rt $\angle$s [*Const*

Hence the circle with centre I_1 and radius I_1D will touch BC and AC and AB produced [*Prop* 37.

EXERCISES

1. Prove that AI_1 bisects the angle A.
2. Prove that $AF = AE = s$
3. Prove that $CD = s - b$ and $BD = s - c$
4. Prove that I, the centre of the inscribed circle, lies on AI_1
5. Prove that the angle ICI_1 is a right angle
6. Prove that the points I, B, I_1, C are concyclic.
7. Prove that the centres of the other two escribed circles lie on I_1B, I_1C produced
8. *If I_1, I_2, I_3 be the centres of the escribed circles opposite to the angles A, B, and C, prove that I_1A, I_2B, I_3C are the altitudes of the triangle $I_1I_2I_3$, and that they all pass through I*
9. *Prove that the radii of the three escribed circles of a triangle are given by*

$$r_1 = \frac{\Delta}{s-a}, \quad r_2 = \frac{\Delta}{s-b}, \quad r_3 = \frac{\Delta}{s-c}$$

10. Prove that $$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r}$$
11. Draw a triangle with sides 13 cm, 14 cm, 15 cm, describe the inscribed and escribed circles, calculate the lengths of their radii and hence verify the formula of the last exercise
12. The distance between the points where the inscribed and escribed circles touch BC is

$$b \sim c$$

Proposition 71 —Problem

Given a regular polygon of any number of sides,

(i) *to describe a circle about it,*

and (ii) *to inscribe a circle in it*

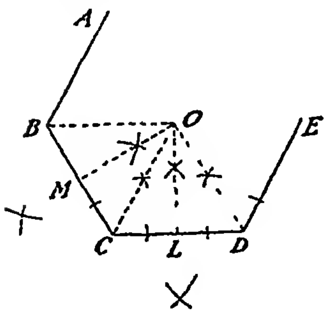
Let AB, BC, CD, and DE be four consecutive sides of a regular polygon

(i) Let the bisectors of the angles at C and D meet in O

Then O is the centre of the circumscribing circle.

Join BO.

Now $\angle OCD = \frac{1}{2}\angle$ of reg pol. $= \angle ODC$,



$\therefore OC = OD$ [*Prop* 12.

Again, in $\triangle$s OCD, OCB,

$$\left\{\begin{array}{l} BC = CD, \quad \text{[\textit{Sides of reg pol.}} \\ CO = CO, \\ \angle OCD = \angle OCB, \quad \text{[\textit{Const.}} \end{array}\right.$$

$\therefore$ $\triangle$s are congruent, [*Prop* 9.

$\therefore OB = OD$

and $\angle OBC = \angle ODC = \frac{1}{2}\angle$ reg pol

Hence $OB = OC = OD$.

Also since BO, CO are bisectors of two adjacent angles, it may be proved as above that $OA = OB = OC$

Thus all lines drawn from O to the angular points of the polygon are equal

Hence O is the centre of the circumscribing circle

(ii) The point O is the centre of the inscribed circle also

Draw OL, $OM \perp CD$, BC respectively.

Then in $\triangle$s OLC, OMC,

$$\left\{\begin{array}{ll} \angle OLC = \angle OMC, & \text{[\textit{Right angles.}} \\ \angle OCL = \angle OCM, & \text{[\textit{Const.}} \\ OC = OC, & \end{array}\right.$$

$\triangle$s are congruent, [*Prop* 10.

and $OL = OM$.

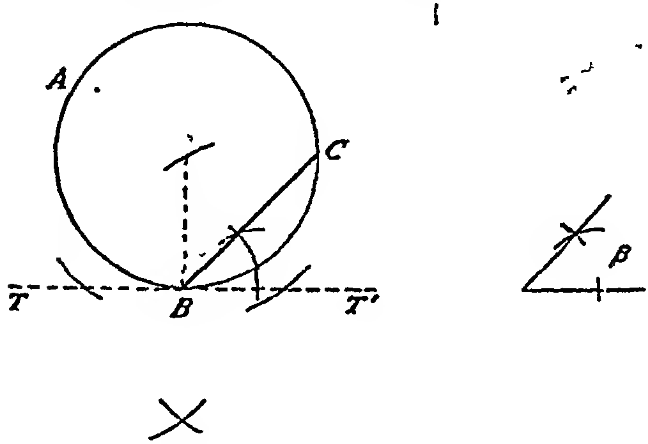
Similarly, all the perpendiculars from O on the sides of the polygon are equal

Hence O is the centre of the inscribed circle.

Proposition 72.—Problem

To cut off from a given circle a segment capable of a given angle

Let ABC be the given circle and β the given angle We are required to cut off from the circle ABC a segment which shall contain an angle equal to β.



At any point B on the circle draw the tangent TBT'. [*Prop* 66

At B make $\angle T'BC = \angle\beta$ [*Prop.* 55.

Then BAC is the required segment

For $\because$ TBT' is a tangent and BC a chord through the point of contact,

$\angle T'BC = \angle$ in alt. segment BAC. [*Prop* 43.

But $\angle T'BC = \angle\beta$. [*Const.*

Hence the angle in the segment BAC is equal to β

EXERCISES

1 *In a given circle inscribe a triangle equiangular to a given triangle.*

Suppose β and γ are the base angles of the given triangle

In the figure of this proposition make the angle $TBA=\gamma$ Then ABC is the required triangle

2 In a circle of 1 5" radius inscribe an equilateral triangle.

3 With sides 1 4", 1 7', and 1 9" draw a triangle, and inscribe a triangle equiangular to it in a circle of 1 2" radius

4. Prove that all triangles, equiangular to a given triangle, which are inscribed in the same circle are congruent

5 In a circle of 3 cm radius inscribe a triangle having two of its angles of 45° and 60° respectively

6 *About a given circle describe a triangle equiangular to a given triangle*

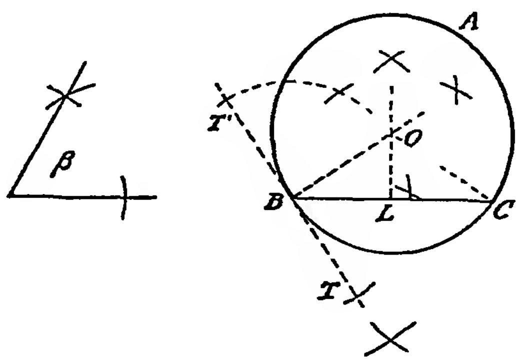
7 About a circle of 1" radius describe a triangle equiangular to the triangle of Ex 3

8 About a circle of 2 cm radius describe an isosceles triangle whose vertical angle is of 30°

9 About a circle of 2 5 cm radius describe a right-angled triangle having an acute angle of 60°

PROPOSITION 73—PROBLEM

On a given straight line to describe a segment of a circle capable of a given angle



Let BC be the given straight line, and β the given angle. It is required to describe on BC a segment of a circle containing an angle equal to β.

Make $\angle CBT = \angle\beta$, [*Prop* 55.

and draw $BO \perp BT$ [*Prop* 53.

Draw LO the right-bisector of BC, meeting BO in O [*Prop* 52.

. O is on the right-bisector of BC,

$OB = OC$ [*Prop* 30.

Hence with O as centre, and OB as radius, describe the circle BAC

Then BAC is the required segment

· $BT \perp BO$, [*Const.*

BT is the tangent at B to the circle, and BC is a chord through the point of contact;

. . $\angle CBT = \angle$ in alt segment. [*Prop* 43.

But $\angle CBT = \angle\beta$ [*Const.*

Hence the segment BAC is capable of the angle β

EXERCISES

1 *On a given straight line describe a segment of a circle which is capable of a right angle*

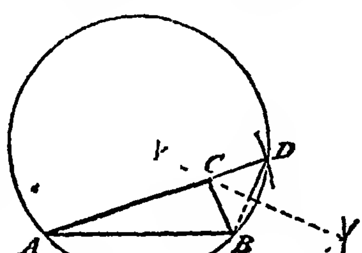
2 Make the construction of this proposition when the given angle is obtuse, say of 135°

3 On a straight line 1 2" long describe a segment of a circle which will contain an angle of 45°

4 On a base 3 cm long describe a segment of a circle capable of an angle of 120°

5 With sides 1 5", 1 7", and 1 9" draw a triangle, find a point within the triangle at which the three sides subtend equal angles

6 *Given the base of a triangle, the sum of the other two sides, and the vertical angle, construct the triangle*



Here c, $a+b$, and C are given

Suppose ABC is the Δ constructed as required

Produce AC cutting off $CD = CB$ Join DB.

Then $CD = CB$, $\angle CDB = \angle CBD$,

$\angle ACB = 2\angle CDB$ [*Prop* 7, *Cor.*

Hence in the triangle ABD the base, the vertical angle D, and the side AD are given

Again, since $CD = CB$, the point C lies on the right-bisector of BD. [*Prop* 30

We have therefore first to construct the triangle ABD, in which

$$AB = c,\ AD = a + b, \text{ and } \angle D = \tfrac{1}{2}\angle C$$

The fixed line AB subtends a constant angle at the point D, therefore the locus of D is the segment of a circle, described on AB as base, which is capable of the angle $\frac{1}{2}C$

Also, the point A is fixed, and the length AD is given, therefore the locus of D is a circle whose centre is A and radius AD

The intersection of these two loci determines the position of D

When the triangle ADB has been constructed we notice that C lies on AD, and also on the right-bisector of BD, hence the position of C is determined

From the above analysis we derive the following construction —

On AB describe a segment of a circle capable of the angle $\frac{1}{2}C$. With centre A, and radius $a + b$, describe an arc cutting this segment in D Join BD.

Draw the right-bisector of BD, cutting AD in C Join BC

Then ABC is the required triangle.

The student ought to complete the proof

7 Construct a triangle, having given the base, the vertical angle, and the altitude

8 Construct a triangle, having given the base, the vertical angle, and the area

9 Construct the triangle ABC, having given $c = 2\ 1''$, $a + b = 3\ 2''$, and $\angle C = 45°$

10 Construct the triangle ABC, having given $c = 1\ 3''$, $C = 60°$, and the perpendicular from C on $AB = 1''$

11 Construct the triangle ABC, having given $c = 1''$, $C = 30°$, and the area of the triangle $= \frac{3}{8}$ sq in

12 Construct a triangle, having given the base, the opposite angle, and the length of the median which bisects the base

13 On a base $1\ 8''$ long construct a triangle whose vertical angle is of $45°$, and whose median bisecting the base is equal in length to the base

14. Construct a triangle, having given the base, the opposite angle, and the difference of the other two sides

16. Construct the triangle ABC, in which $c=3.5''$, $a-b=1''$, and $C=60^\circ$.

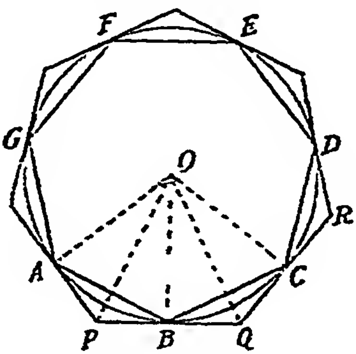
PROPOSITION 74.—PROBLEM

In or about a given circle to construct a regular rectilineal figure of any number of sides.

DEF.—A polygon of n sides is called an n-gon.

(i.) In a given circle to inscribe a regular n-gon.

Divide (if possible) the circumference into n equal arcs at the points A, B, C, D, E, F, G.



Then the chords AB, BC. CD . . . will form the inscribed regular n-gon

$\because$ arcs AB, BC . . . are equal, [*Constr.*

$\therefore$ chords AB BC . . . are equal. [*Prop.* 35

Hence the n-gon is equilateral.

Again, arc AB = arc CD;

$\therefore$ arc AB − arc BC = arc BC − arc CD;

i e arc ABC = arc BCD;

$\therefore \angle ABC = \angle BCD$.

[*Angles in equal segments.*

In the same way it may be shown that all the angles of the n-gon are equal.

Hence the n-gon is equiangular.

But we have already shown it to be equilateral

Therefore the n-gon is regular

(ii) About a given circle to circumscribe a regular n-gon

At the angular points of the inscribed regular n-gon draw tangents PQ, QR . . ., then these tangents will form the circumscribed regular n-gon

Each side of the inscribed n-gon subtends the same angle at the centre, viz $\dfrac{360°}{n}$

Therefore $\angle AOB = \angle BOC = .$

Now, in Δs BOQ, COQ,

$$\cdot \begin{cases} OB = OC, & \text{[Radii} \\ QB = QC, & \text{[Two tangents from a point} \\ OQ = OQ, & \end{cases}$$

Δs are congruent, [Prop 13.

and $\angle BOQ = \angle COQ$

$$= \tfrac{1}{2}\angle\frac{360°}{n}.$$

Similarly, $\angle AOP = \angle BOP$

$$= \tfrac{1}{2}\angle\frac{360°}{n};$$

$$\angle BOP = \angle BOQ.$$

Again, in Δs BOP, BOQ,

$$\cdot \begin{cases} \angle BOP = \angle BOQ, & \text{[Proved} \\ \angle OBP = \angle OBQ, & \text{[Right angles} \\ OB = OB, & \end{cases}$$

Δs are congruent, [Prop 10.

and $BP = BQ$

In the same way it may be proved that

$$CR = CQ$$

Therefore $PQ = 2BQ$ and $QR = 2CQ$.

But $BQ = CQ$, being two tangents from the same point to a circle.

Therefore $PQ = QR = \ldots$

Thus the n-gon $PQR \ldots$ is equilateral.

We have also $\angle PQR$ = supplement of $\angle BOC$,

$$= \text{supplement of } \angle\frac{360^\circ}{n}.$$

Thus all the angles $P, Q, R \ldots$ are supplements of the same angle, and are therefore equal

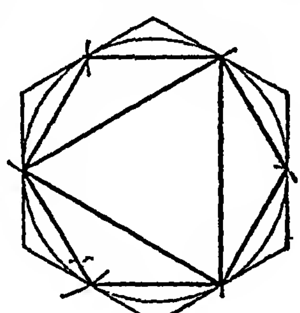
Hence the n-gon $PQR \ldots$ is both equilateral and equiangular

Note—The constructions given above can only be effected in those cases in which it is possible to divide the circumference of a circle into a number of equal parts Some of these important cases are given in the exercises which follow.

EXERCISES

1 *In a given circle inscribe a regular hexagon*

If a chord equal to the radius be placed in a circle, the angle subtended by the chord at the centre of the circle is clearly equal to two-thirds of a right angle It follows therefore that if six chords, each equal to the radius, be placed end to end in a circle, the sum of the angles subtended by them at the centre of the circle is equal to four right angles Hence six chords so placed will completely go round the circumference and form the sides of the inscribed hexagon



2 *Circumscribe a regular hexagon about a given circle*

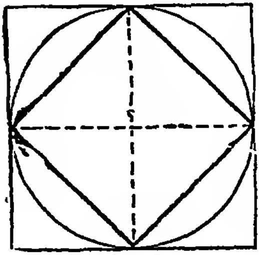
3 *Inscribe an equilateral triangle in a given circle*

Join the alternate angular points of the inscribed hexagon

4 *Circumscribe an equilateral triangle about a given circle*

5 Show that the area of the inscribed equilateral triangle is equal to half the area of the inscribed regular hexagon

6 The area of the inscribed regular hexagon is three-fourths of the area of the circumscribed regular hexagon



7 Show that the area of the circumscribed regular hexagon is two thirds of the area of the circumscribed equilateral triangle

8 An equilateral triangle and a regular hexagon are inscribed in the same circle, show that the square on the side of the triangle is three times the square on the side of the hexagon

9 *Inscribe a square in a given circle*

Two perpendicular diameters will divide the circumference into four equal parts

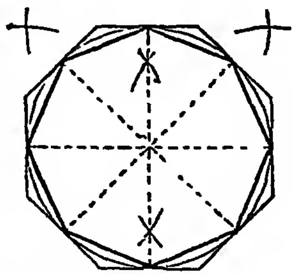
10 *Circumscribe a square about a given circle*

11 Show that the inscribed square of a circle is half the circumscribed square.

12 Show that the square described on the radius of a circle is half the inscribed square

13 *Inscribe a regular octagon in a given circle*

Draw the bisectors of the angles between two perpendicular diameters. These four lines will divide the circumference into eight equal parts



14. *Circumscribe a regular octagon about a given circle*

15 *Show that when a regular n-gon has been inscribed in a circle a regular 2n-gon can be inscribed by simply bisecting the arcs*

16 *In and about a given circle construct a regular dodecagon*

17 In and about a circle of 1″ radius describe an equilateral triangle.

18 Inscribe a regular hexagon in a circle of 3 cm radius

19 In and about a circle of 8″ radius describe a square

20 In and about a circle of 1 3″ radius construct a regular octagon

21 In a circle of 1 2″ radius place two co terminous chords which are respectively equal to the sides of an inscribed equilateral triangle and a square.

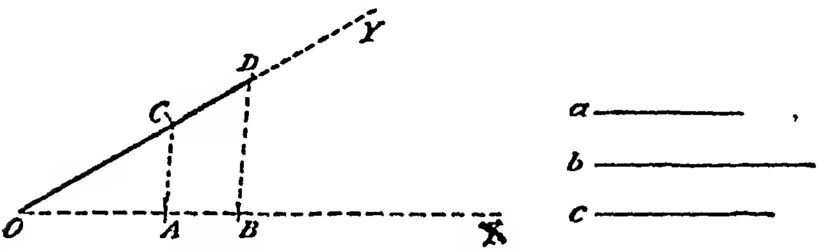
22 In a circle of 5 cm radius inscribe an equilateral triangle, a regular hexagon, and a regular dodecagon

23 In a circle of 4 5 cm radius inscribe a square and a regular octagon; also construct the circumscribing regular octagon.

PROPOSITION 75.—PROBLEM

To construct a fourth proportional to three given straight lines.

Let a, b, c be three given straight lines.



It is required to construct a fourth line x, such that

$$\frac{a}{b}=\frac{c}{x}.$$

Take OX, OY any two lines, making an angle

On OX step off OA, OB equal to a and b respectively; and on OY step off OC equal to c

Join AC.

Draw $BD \parallel AC$, meeting OY in D

Then $$OD=x.$$

For in $\triangle OAC$ the straight line BD is drawn parallel to AC,

$$\therefore \frac{OA}{OB}=\frac{OC}{OD}, \qquad \text{[Prop. 45.}$$

i.e. $$\frac{a}{b}=\frac{c}{OD}.$$

Hence OD is the required fourth proportional.

EXERCISES

1 *To find a third proportional to two given lines.*

Let a, b be the two given lines It is required to find a third line x, uch that

$$\frac{a}{b}=\frac{b}{x}.$$

Take two lines OX, OY as above

On OX set off $OA=a$, $OB=b$, and from OY cut off $OC=OB=b$. Join AC, and draw BD parallel to AC Then OD is the required third proportional

2 Find by construction a fourth proportional to three lines whose lengths are 1 7″, 2 3″, and 2 9″.

3 Find by geometrical construction the value of $\frac{1\ 9\times 3\ 1}{1\ 7}$, taking 1 cm as unit

4. Measure a line 3 3″ long, and divide it in the ratio of 3 8

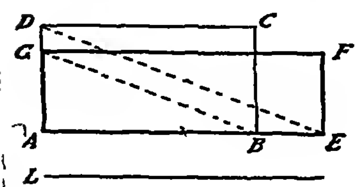
5 Divide a line 4″ long in the ratio of 2 3 3

6 Construct a triangle whose perimeter is 6″, and whose sides are in the ratio of 3 4 5

7 Find by geometrical construction the value of $\frac{(3\ 7)^2}{2\ 5}$, taking 1″ as the unit of length

8 *Construct a rectangle equal in area to a given rectangle, and having one side equal to a given straight line*

Let $ABCD$ be the given rectangle and L the given straight line



From AB, produced if necessary, cut off AE equal to L.

Join ED

Draw $BG \parallel ED$

Complete the rectangle $AEFG$

In ΔAED, $BG \parallel ED$,

$$\therefore \frac{AB}{AE}=\frac{AG}{AD},$$

i e $AE\ AG=AB\ AD$

Hence we have constructed a rectangle of area $AB\ AD$, and having one side equal to L

9 Construct a rectangle containing an area of 2 sq in, and having a side 2 9″ long

10 Draw a parallelogram with sides 2″ and 3″, and included angle of 45°, on a base 2 3″ long construct a rectangle of equivalent area.

11 Draw a rhombus of 5 cm side, and having an angle of 60°, construct a rectangle of equivalent area, and having one of its sides 3 7 cm long

12 Construct a rectangle having a side 2 3″ and the same area as that of a triangle whose sides are 1 8″, 2″, and 2 2″

13 Construct a parallelogram containing an area of 2 sq in, and having adjacent sides 1 5″ and 2 5″

14 Construct a triangle, one side 2 2″, area 3 sq in, and one angle of 60°

15 Given two intersecting lines and a point lying between them, through the given point draw a straight line so that the segments intercepted between the point and the given lines may be in the ratio of 3 5

16 Given two intersecting lines and a point lying outside the angle formed by them, through the point draw a straight line so that the segments intercepted between the point and the given lines may be in the ratio of 1 3

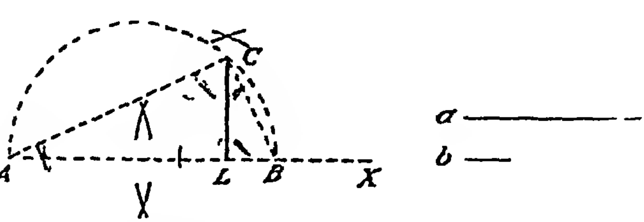
17 Construct a rhombus, having the same area as an equilateral triangle of 2″ side, and having one of its angles equal to an angle of an equilateral triangle

Proposition 76 —Problem

To construct a mean proportional to two given straight lines

Let a, b be the two given straight lines It is required to construct a straight line x such that

$$\frac{a}{x}=\frac{x}{b}, \text{ or } x^2=ab.$$



From any straight line AX cut off $AL=a$ and $LB=b$ On AB describe the semicircle ACB.

Draw $LC \perp AB$ to meet the circumference in C.

Then LC is the required mean proportional.

ACB is a semicircle,

$\angle ACB$ is a rt. $\angle$. [*Prop* 41.

In the right-angled triangles ALC, BLC,

$\angle CAL =$ complement of $\angle ACL = \angle BCL$,

$\triangle$s are equiangular.

Hence
$$\frac{AL}{LC} = \frac{LC}{LB},$$

$$LC^2 = AL \cdot LB$$

Therefore LC is a mean proportional to AL and LB, i e to a and b

EXERCISES

1 Find by construction a mean proportional between two lines whose lengths are 4″ and 1′, and check by measurement

2 Find by geometrical construction a mean proportional to two lines of lengths 4 5 cm and 2 cm respectively

Check the correctness of the drawing by measurement

3 Taking one centimetre as the unit of length construct the lines whose lengths are $\sqrt{(5)}$ and $\sqrt{(6)}$ respectively.

4 If x, y, z, u be the lengths of four given lines, show how to construct the line whose length is

$$\sqrt[4]{(xyzu)}$$

5 Taking 1″ as the unit of length construct the line whose length is

$\sqrt[4]{30}$ inches

6 *From any straight line OX cut off $OA = a$, $OL = b$, on OA describe a semicircle, and draw LC at right angles to OA to meet the semicircle in C, join OC*

Prove that OC is a mean proportional to a and b

Note —In practical geometry this construction is required as often as the one given in the text

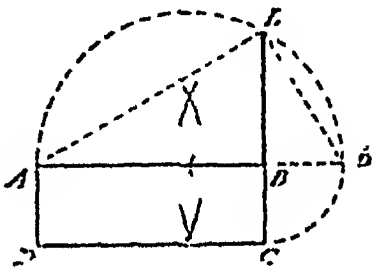
PROPOSITION 77.—PROBLEM

To construct a square equal in area to a given polygon.

We can construct a triangle equal to the given polygon (Prop. 64) and then construct a rectangle equal to this triangle (Prop. 65).

Thus the problem is reduced to—

The construction of a square equal in area to a given rectangle.



Let $ABCD$ be the rectangle.

Produce AB, and cut off BP equal to BC.

On AP as diameter describe the semicircle APL.

Produce CB to meet the circumference in L.

Then BL is a side of the required square.

The $\angle ALP$ is a rt. $\angle$. [*Prop.* 41.

In the right-angled triangle ALP, LB is drawn a perpendicular from the right angle on the hypotenuse;

$\therefore\ LB^2 = AB \cdot BP.$ [*Prop.* 76.

But $BP = BC$; [*Const.*

$\therefore\ LB^2 = AB \cdot BC.$

EXERCISES

1. Draw a rectangle with sides 2 cm. and 8 cm.; construct a square equal to it in area.

2. Construct a square equal in area to a rectangle whose sides are 5″ and 1.8″. Check by calculation and measurement.

3. Draw a parallelogram with sides 1.7″ and 2.3″, and included angle of 30°; construct a square of equivalent area.

4 Construct a square equal in area to a triangle whose sides are 1 9", 2 3", 2 5"

5 Construct a square equal in area to a rhombus whose side is 6 cm , and one of whose angles is of 60°

6 Given one side of a rectangle which is equal in area to a given square, find the other side

7 Draw the quadrilateral *ABCD*, in which *AB*=5", *BC*=2"=*CD*, *AD*=3", and *AC*=4" , construct a square equal to it in area

8 Construct a right angled triangle equal in area to a given polygon, and having one of the sides containing the right angle twice as great as the other

9 Construct a right-angled isosceles triangle equal to a given polygon

MISCELLANEOUS EXERCISES —VI

1 To a circle of 1 5" radius draw a tangent from a point distant 2 5" from its centre Calculate the length of the tangent

2 Through a point distant 3 7" from the centre of a circle of radius 1 2" draw a pair of tangents to the circle, and find their lengths

3 In the last exercise find the distance of the chord of contact from the centre of the circle

4 Find a point outside a circle such that the chord of contact of tangents drawn from it to the circle is equal to the central distance of the point

5 Find a point outside a given circle from which tangents drawn to the circle contain a given angle

Find also the locus of all such points

6 Find a point outside a given circle from which a tangent is equal to the diameter of the circle.

7 Draw two circles of radii 1 cm and 4 cm , touching one another externally , find the lengths of their common tangents

8 If two circles touch externally, one of their common tangents bisects each of the other two

9 Draw two circles with radii 6 cm and 11 cm and centres 13 cm apart , draw their direct common tangents and calculate their lengths

10 Draw two circles with radii 2 1 cm and 4 2 cm and centres 6 5 cm apart , construct their common transverse tangents and calculate their lengths

11 In an equilateral triangle the circumcentre and incentre are coincident, and the circumradius is double of the inradius

12 What is the simplest construction for circumscribing a circle about a right-angled triangle?

13 Prove that if the incentre and circumcentre of a triangle coincide the triangle is equilateral

14. *The angle between the circum-radius to the vertex of a triangle and the altitude through the vertex is equal to the difference of the base angles of the triangle.*

15 If the join of the incentre and circumcentre of a triangle pass through an angular point the triangle is isosceles

Intersection of Loci.—When a point is required to satisfy two given conditions its position may be determined in the following way —

First find the locus of the point on the assumption that it satisfies the first condition only.

Next find the locus of the point on the assumption that it satisfies the second condition only.

Then the points of intersection of the loci will satisfy each of the given conditions

We have already had examples of this method, and a few more will be considered here.

16 *Find a point which shall be at given distances r_1 and r_2 from two fixed points A and B respectively*

The locus of a point which is at a distance r_1 from A is a circle whose centre is A and radius r_1

The locus of a point which is at a distance r_2 from B is a circle whose centre is B and radius r_2.

Hence the required point is determined by the intersection of these two circles.

Here there are three cases to be considered —

(i.) When $r_1 + r_2 > AB$

The circles will intersect in two points, hence there are *two solutions* to the problem.

(ii.) When $r_1 + r_2 < AB$

The circles will not intersect, hence there is *no solution.*

(iii.) When $r_1 + r_2 = AB$

The circles will touch at one point, hence there is *one solution* only

Thus with the given data there may be two solutions, one, or none, according as the sum of the given distances is greater than, equal to, or less than the distance between the given points

17 *Construct a triangle, having given the area, the base, and the median bisecting the base*

Since the base is given we are required only to find the position of the vertex

The base and area being given the altitude is known [*Prop* 21, *Cor* 2

When the altitude and base of a triangle are given, the locus of the vertex is two straight lines parallel to the base, on either side of the base, and at the distances of the altitude from the base

When the median bisecting the base of a triangle is given, the locus of the vertex is a circle, whose centre is at the middle point of the base, and whose radius is equal to the given median

Hence the position of the vertex is determined by the points of intersection of the two parallels and this circle

By drawing a figure the student will find that there are three cases to be considered —

(i) When the median > the altitude there are four solutions

(ii) When the median < the altitude there is no solution

(iii) When the median = the altitude there are two solutions and both the triangles are isosceles

18 *Find a point equidistant from three given straight lines forming a triangle*

Let X, Y, Z be the three given straight lines

The locus of a point equidistant from X and Y is the pair of lines which bisect the angles between X and Y [*Prop* 31

The locus of a point equidistant from Y and Z is the pair of lines which bisect the angle between Y and Z

Hence the point equidistant from X, Y, and Z is determined by the intersections of these two pairs of lines

Now two pairs of lines will intersect in four points, therefore there are four points which are equidistant from the three given lines, viz. the *incentre* and the *three excentres* of the triangle formed by the lines

*19 Describe a circle which shall touch two given straight lines and pass through two given points

*20 Describe a circle which shall pass through two given points and touch a given circle at a given point

21 Construct an isosceles triangle, being given the base and vertical angle.

---

* In 19 and 20 find the conditions that a solution may be possible.

22 Find a point at given distances from two intersecting straight lines. Show that there are four such points, and that they form a parallelogram

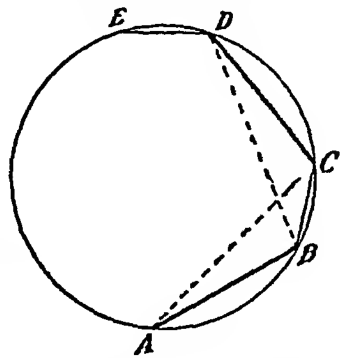
23 Draw the triangle ABC, having given $a=5$ cm, $b=4\ 5$ cm, $c=4$ cm, find a point which is equidistant from B and C, and which is at a distance of 2 4 cm from A

24. Draw the triangle ABC in which $a=2\ 9''$, $b=2\ 3''$, $c=3\ 5''$, find a point equidistant from AB and AC and at a distance of 1 2″ from BC

Show that there are four solutions

25 *An equilateral rectilineal figure inscribed in a circle is also equiangular*

26 *An equiangular rectilineal figure inscribed in a circle has its alternate sides equal*



In Δs ABC, BCD,

$$\therefore \begin{cases} \angle ABC=\angle BCD, & \text{[Hyp} \\ \angle BAC=\angle BDC, & \text{[Prop 40} \\ BC=BC, & \end{cases}$$

$\therefore$ Δs are congruent, [*Prop* 10

and $AB=CD$

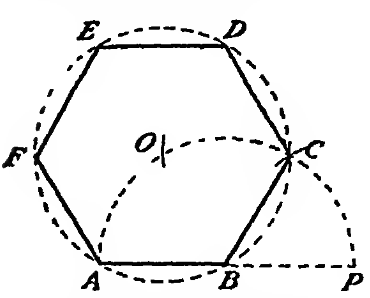
Similarly, $BC=DE$

27. *An equiangular rectilineal figure with an odd number of sides inscribed in a circle is also equilateral*

28 *An equiangular rectilineal figure described about a circle is equilateral*

29 *An equilateral rectilineal figure described about a circle is equiangular if the number of sides be odd*

30 *On a given finite straight line construct a regular hexagon*



Let AB be the given straight line

Produce AB and cut off $BP=AB$ On AP describe a semicircle and divide its arc into three equal parts by placing chords PC, CO equal to the radius

Then BC is a side of the hexagon and O is the centre of its circumscribing circle

The figure can be completed by placing chords equal to AB in the circumference of the circle ABC.

31 On a straight line 3 cm long describe a regular hexagon, and prove that six of its diagonals are parallel in pairs

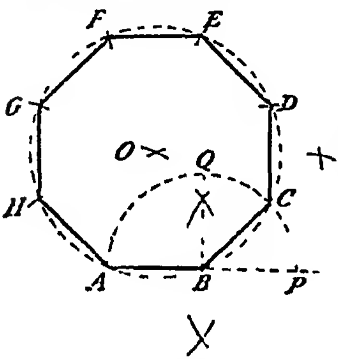
32 On a straight line 1 6" long draw a regular hexagon, and inscribe a circle in it

33 On a straight line 3" long construct a regular hexagon, join the alternate angular points, and prove that the joins enclose another regular hexagon

Inscribe a circle in the smaller hexagon

34 Show how to cut out a regular hexagon from a given equilateral triangle

35 *On a given finite straight line construct a regular octagon*



Let AB be the given straight line

Produce AB and cut off $BP = BA$

On AP describe a semicircle, and divide its arc into four equal parts by bisecting the arc AP in Q and the arc QP in C

Join BC

Then BC is a side of the octagon

For $\angle CBP = \frac{1}{2}$ (right angle) [*Const*

$=$ external $\angle$ of reg octagon

Draw the circumcircle of the triangle ABC, and step off the chord AB round the circumference

36 On a straight line 1 4" long construct a regular octagon, and inscribe a circle in it

37 Construct a regular octagon of 4 cm side, join the alternate angular points, and prove that the joins enclose another regular octagon.

38 Show how to cut out of a given square a regular octagon

39 On a given straight line describe a regular dodecagon

40 Describe a regular dodecagon of 3 cm side, and in it inscribe a circle

41. On a straight line 1 8" long describe a regular dodecagon, join the alternate angles to form a regular hexagon, and join the alternate angles of the hexagon to form an equilateral triangle

42 With the notation of the exercises in Props. 69 and 70 prove that

$$r\ r_1\ r_2\ r_3 = \Delta^2.$$

43 Prove that in the figure of Prop 70 the triangles I_1BC, I_2CA, I_3AB are equiangular

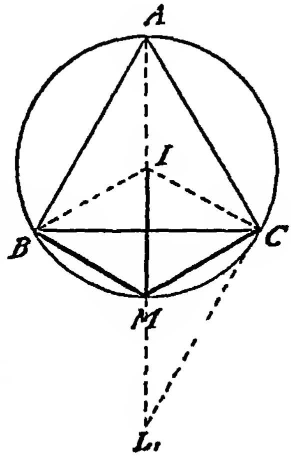
44 Construct a triangle, having given the centres of its three escribed circles

45 Construct a triangle, having given two of its excentres and the incentre

46 *In the triangle ABC the bisector of the vertical angle meets the circumcircle of the triangle again in M*
Prove that

$$MB = MC = MI,$$

where I is the incentre



47 Given the base and vertical angle of a triangle, find the locus of its incentre

48 Given the base and vertical angle of a triangle, find the locus of its excentre opposite the given vertical angle

49 Describe three circles, each touching two sides of a given triangle and its incircle

50 Draw a triangle with sides 4 2", 3", and 4 5", construct three circles, each touching two sides of the triangle and its incircle

51 Construct a rhombus, having given the radius of its inscribed circle and one of its angles

52 With the usual notation prove that

$$r r_1 = (s-b)(s-c)$$

53 In the figure of Prop 70 prove that

$$AI \cdot AI_1 = bc,\ BI \cdot BI_2 = ca,\ CI \cdot CI_3 = ab$$

54 Inscribe a circle in a given sector of a circle

55 Construct a triangle, having given the vertical angle and both the inradius and circumradius

56 Construct a triangle, having given the vertical angle, the base, and the inradius

57 Construct a triangle, having given the base, the vertical angle, and the exradius corresponding to the given vertical angle

58 Describe three circles touching each other externally and having their centres at the angular points of a given triangle

59 Inscribe three circles in an equilateral triangle, touching each other, and each of them touching two sides of the triangle

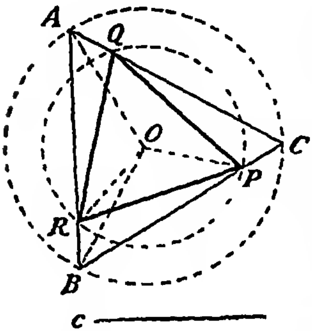
60 Inscribe three circles in an equilateral triangle, touching each other, and each of them touching one side of the triangle.

61 Prove that the inradius of a triangle is less than any one of the three exradii

62 Construct a triangle, having given the base, the vertical angle, and the perpendicular from one extremity of the base on the opposite side

63 Construct a triangle, having given the base, the vertical angle, and the point where the bisector of the vertical angle meets the base

64 Find the locus of points from which tangents drawn to a given circle are at right angles to each other



65 *In a given equilateral triangle inscribe another equilateral triangle having each side equal to a given straight line*

Let ABC be the given equilateral triangle, and PQR the inscribed triangle, having each side equal to the given line c Let each side of the given triangle be equal to a

Draw the circumcircles of the two triangles, and let their radii be R and r A little consideration will show that these circles must be concentric.

In ΔAOB

$$a^2 = 3R^2, \qquad \text{[Prop 68, Exs.}$$

$$\therefore a = \sqrt{3}\ R$$

In ΔPOR

$$c^2 = 3r^2,$$

$$c = \sqrt{3}\ r,$$

$$a : c = R : r$$

Thus r is a fourth proportional to three known lines a, c, and R.

Hence the construction

66 In a given square inscribe another square having each side equal to a given line

67 In a given hexagon inscribe another hexagon having each side equal to a given line

68 Describe six equal circles, each touching a given circle and two of the six.

69 Describe three equal circles to touch each other and a given circle

70 Prove that a rectilineal figure which is described about one and inscribed in another of two concentric circles must be regular

BOOK III

MISCELLANEOUS PROPOSITIONS

In this book we shall discuss certain elementary propositions which are not included in the syllabus for the Matriculation Examination, but are of importance to the student, as they contain a few fundamental facts in geometry with which he ought to become familiar at an early stage in his mathematical studies

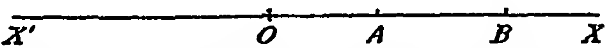
Except in the case of the simplest propositions, we shall generally give short outlines of proofs, and we shall expect that the student, with the help of the teacher, will write out full proofs for himself.

SECTION I—SEGMENTS OF A LINE

In order that we may be enabled to use the formulæ of Algebra in investigating geometrical theorems it is necessary that we should regard a length in geometry to possess both **magnitude** and **sign**

Consider an indefinite line $X'OX$ extending on both sides of a fixed point O, which we shall call the **origin**

If lengths measured from O in the direction OX be considered positive, then those measured from O in the opposite direction OX' are considered negative



Again, if AB, the segment measured from A to B, be considered positive, then BA, the segment measured from B to A, is considered negative

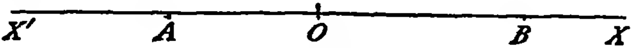
Hence, using the symbols of algebra, we have

$$AB = -BA \qquad (\text{i.})$$

It is clear from the above figure that

$$AB = OB - OA \qquad (\text{ii.})$$

A little consideration will show that this formula is true for all positions of the points A and B on the line $X'OX$



For $\quad OB - OA = OB + AO$ [*By* (i)

$\qquad = AO + OB$ [*Changing order of terms*

$\qquad = AB$

The whole theory of a divided line is based on the formulæ (i) and (ii), the student ought to satisfy himself of their universal truth by taking the points A and B in all possible positions relatively to the fixed origin O

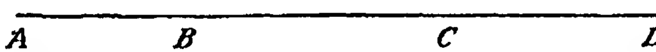
DEF —*Points which lie on the same straight line are said to be* collinear

PROPOSITION I

Euler's Theorem

If A, B, C, D be any four collinear points, then

$$AB\ CD + AD\ BC + AC\ DB = O.$$



Take A as origin, and let $AB = b$, $AC = c$, $AD = d$

Then $CD = d - c$, $BC = c - b$, $DB = -(d - b)$.

Hence $\quad AB\ CD + AD\ BC + AC\ DB = b(d - c)$

$$+ d(c - b) - c(d - b)$$

$$= bd - bc + dc - db - cd + bc$$

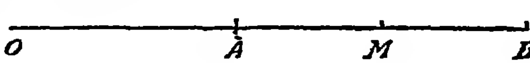
$$= O$$

PROPOSITION II

If M be the middle point of the line AB and O any other point on the line, then

$$OM = \tfrac{1}{2}(OA + OB)$$

Take O as origin .



We have $$AM=MB, \qquad [Hyp.$$

$$\therefore OM-OA=OB-OM \qquad [By (ii).$$

Hence $$OM=\tfrac{1}{2}(OA+OB).$$

EXERCISES

1 If A, B, C be three collinear points, then

$$BC+CA+AB=0$$

2 If A, B, C, D be four collinear points, then whatever be the order in which they are situated—

$$AB+BC+CD+DA=0$$

3 If A, B, C, D, E be five collinear points, then whatever be the order in which they are situated—

$$AB+BC+CD+DE+EA=0$$

4 If A, B, C be three collinear points, and O any point whatever, then

$$\Delta OBC+\Delta OCA+\Delta OAB=0,$$

regard being had to the signs of the areas as well as their magnitudes

5 If A, B, C, D be four collinear points, and O any point whatever, then

$$\Delta OAB.\Delta OCD+\Delta OAD.\Delta OBC+\Delta OAC.\Delta ODB=0,$$

regard being had to the signs of the areas as well as their magnitudes.

6 In the figure of Prop II prove that—

$$\text{(i.) } OA^2+OB^2=AM^2+BM^2+2OM^2,$$

$$\text{(ii.) } OA^2-OB^2=2AB.MO,$$

$$\text{(iii.) } OA.OB=OM^2-AM^2,$$

$$\text{(iv.) } OA^2+OB^2=AB^2+2OA.OB$$

7 If from any point O a perpendicular OL be drawn to the line AB, then

$$OA^2-OB^2=AB^2+2AB.BL$$

8 If CP be any line drawn from the vertex to the base of the triangle ABC, then

$$AP.CB^2-BP.CA^2=AB(CP^2-AP.BP)$$

9 If A, B, C be three collinear points, and P any other point whatever, then

$$AP^2.BC+BP^2.CA+CP^2.AB+BC.CA.AB=0.$$

10 If A, B, C, P be four collinear points, then

$$AP^2.BC+BP^2.CA+CP^2.AB+BC.CA.AB=0$$

Medial Section

DEF —*When a straight line is divided into two segments such that the rectangle contained by the whole line and one of the segments is equal to the square on the other segment, the line is said to be divided in* **medial section.**

Thus AB is said to be divided in medial section at the point P if

$$AB \cdot PB = AP^2$$

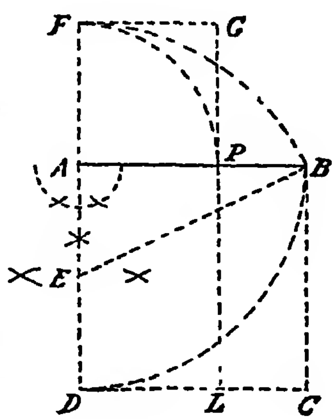
As a consequence of this relation we have

$$\frac{AB}{AP} = \frac{AP}{PB},$$

i e the ratio of the whole line to one segment is equal to the ratio of that segment to the other

Hence the line is also said to be divided in **extreme and mean ratio.**

PROPOSITION III



To divide a given straight line in medial section

Let AB be the given straight line On AB describe the square $ABCD$

Bisect AD in E, and from DA produced cut off EF equal to EB On AF construct the square $AFGP$.

Then AB is divided in P as required

Produce GP to L

Now rect. $FL = DF \cdot FA$

$$= (EF + EA)(EF - EA)$$
$$= EF^2 - EA^2$$
$$= EB^2 - EA^2 \qquad \text{[Const}$$
$$= AB^2 \qquad \text{[Bk I, Prop 27.}$$

From each of these equals take away the rect AL,

then $$\text{fig } FP = \text{fig } PC,$$

i e $$AP^2 = AB \cdot PB$$

EXERCISES

1 Prove that DF is divided in medial section at A

2 Prove that FG, CB, and DP meet in the same point O.

3 Prove that DO is divided in medial section at P

4 Prove that

$$AB^2 + PB^2 = 3AP^2$$

5 Prove that

$$(AB + PB)^2 = 5AP^2$$

6 Prove that

$$AP^2 - PB^2 = AP \cdot PB$$

7 From PA cut off PQ equal to PB, prove that AP is divided at Q in medial section

8 Prove that DP is perpendicular to BF

9 Prove that LF is parallel to CP

10 Prove that

$$AB^2 - AP^2 = AB \cdot AP.$$

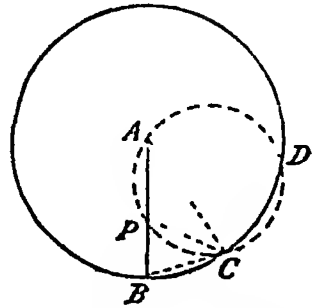
PROPOSITION IV

If the radius of a circle be divided in medial section, the greater segment is equal to a side of a regular decagon inscribed in the circle

Let AB be the radius of a circle whose centre is A, and let AP be divided at P in medial section, so that

$$AP^2 = AP \cdot PB,$$

then AP is equal to a side of a regular decagon inscribed in the circle A



Place in the circle a chord BC equal to AP

Join CP, AC

Draw the circumcircle of the triangle APC

Then $AP \cdot PB = AP^2$ [*Hyp.*
$= BC^2$, [*Const.*

$\therefore$ BC is a tangent to the circle APC [*Prop* 44, *Cor.* 2.

Now $\angle BCP = \angle CAP$ in the alternate segment [*Prop* 43

And $\angle CPB = \angle CAP + \angle PCA$ [*Prop* 7, *Cor.*
$= \angle BCP + \angle PCA$
$= \angle BCA$
$= \angle ABC$, [*Prop* 11.

$\therefore$ $\angle CPB = \angle CBP$,
$CB = CP$ [*Prop* 12
$= AP$. [*Const*

Again, $\because$ $CP = AP$,
$\angle CPB = 2\angle CAP$ [*Props* 11 and 7, *Cor.*

Therefore $\angle CBA = 2\angle BAC$

Thus *we have constructed an isosceles triangle ABC, having a given side AB, and having each of the angles at the base double of the angle at the vertex.*

Since all the angles of the triangle ABC are together equal to two rt $\angle$s,

$$\angle BAC = \tfrac{1}{5} \text{ of } 2 \text{ rt } \angle\text{s}$$
$$= \tfrac{1}{10} \text{ of } 4 \text{ rt } \angle\text{s}.$$

Hence BC or AP is equal to a side of a regular decagon inscribed in the circle whose radius is AB.

EXERCISES

1 In a circle of 1 4″ radius inscribe a regular decagon

2 *Inscribe a regular pentagon in a given circle*

3 In a circle of 3 2 cm radius inscribe a regular pentagon.

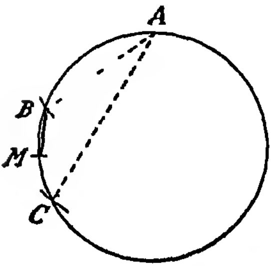
4 *About a circle of 1″ radius describe a regular pentagon.*

5 Construct angles of 36°, 72°, 54°, 18°, and 108°

6 *Divide a right angle into five equal parts*

7 Divide a right angle into fifteen equal parts

8 *In a given circle inscribe a regular quindecagon*



Let AB be a side of an inscribed regular pentagon, and AC a side of an inscribed equilateral triangle

Bisect the arc BC in M, and join BM; then BM is a side of an inscribed regular quindecagon

For arc $AC = \frac{1}{3}$ circumference,
and arc $AB = \frac{1}{5}$ circumference,
arc $BC = \frac{2}{15}$ circumference,
and arc $BM = \frac{1}{15}$ circumference

9 *On a given base describe an isosceles triangle, having each of the angles at the base double of the vertical angle*

10 In the figure of this proposition prove that—

(i) $\angle APC = 3\angle PAC = 3\angle PCA$

(ii) CP is the side of a regular pentagon inscribed in the circle APC

11 On a given straight line as base describe an isosceles triangle having each of the angles at the base one-third of the vertical angle

12 *On a given base describe a regular pentagon*

13 Describe a regular pentagon having each side 2″ long

14 In the figure of this proposition, if D be the other point in which the circle APC cuts the circle A, then prove that—

(i) Δs ACD, ABC are equiangular,

(ii) AD is parallel to CP,

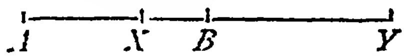
(iii) PD is parallel to BC,

(iv) PD is equal to AB

15 *If regular polygons of five, six, and ten sides be inscribed in a circle, then the square on the side of the pentagon is equal to the sum of the squares on the sides of the hexagon and decagon* (Euclid, bk. xiii. 10)

Harmonic Section

Def.—*A line AB is said to be divided* **harmonically** *at two points X and Y when the ratios* $\frac{AX}{BX}$ *and* $\frac{AY}{BY}$ *of the two pairs of segments into which it is divided are equal in magnitude and opposite in sign.*



The line AB is divided at X *internally* in the ratio of $AX : BX$, and *externally* in the ratio of $AY : BY$.

The second term of the first ratio is negative, but both terms of the second ratio are positive.

Notice also that according to the fundamental formula (ii.), the difference of the segments is in each case equal to the length of the given line.

Thus $AX - BX = AB$ and $AY - BY = AB$.

Def.—*The points X and Y are called* **harmonic conjugates** *with respect to the points A and B, and AXBY is called a* **harmonic range**.

EXERCISES

1. If X, Y are harmonic conjugates with respect to A, B, then A, B are harmonic conjugates with respect to X, Y.
2. If $AXBY$ be a harmonic range, prove that—

$$\text{(i.) } \frac{2}{AB}=\frac{1}{AX}+\frac{1}{AY}; \quad \text{(ii.) } \frac{2}{XY}=\frac{1}{XA}+\frac{1}{XB}.$$

3. If M be the middle point of XY, then $AX \cdot AY = AB \cdot AM$.
4. If A, A' and B, B' be harmonic conjugates, prove that—

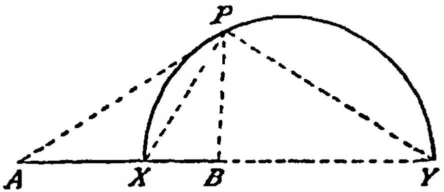
$$\frac{1}{A'B}+\frac{1}{A'B'}=\frac{2}{A'A}$$

5. *The internal and external bisectors of the vertical angle of a triangle divide the base harmonically.*

Proposition V

The Apollonian Locus

The locus of a point, the ratio of whose distances from two fixed points is constant, is a circle



Let A, B be the fixed points, and let P be any point on the locus

Join PA, PB

Let the bisectors of the internal and external angles at P meet AB in X, Y Then

$$\frac{AP}{BP} = \text{constant}, \qquad [Hyp.$$

$$\frac{AX}{BX} = \frac{AY}{BY} = \text{the same constant}$$

[*Bk I*, *Prop* 49

Therefore X and Y are the harmonic conjugates of A and B, and are consequently fixed points

Again, $\angle XPY$ is a rt $\angle$

Hence the locus of P is a circle described on XY as diameter

EXERCISES

1 Find a point whose distances from three given points shall be to one another in the ratio of three given lengths

2 Draw an equilateral triangle of 3″ side, and find a point whose distances from the angular points of the triangle are in the ratio of 2 : 3 4

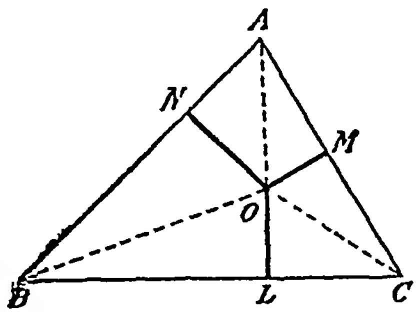
DEF —*A set of lines which pass through the same point are said to be* **concurrent.**

PROPOSITION VI

If three concurrent lines LO, MO, NO intersect the sides BC, CA, AB of the triangle ABC at right angles in L, M, N, then the segments of the sides satisfy the relation

$$(BL^2 - CL^2) + (CM^2 - AM^2) + (AN^2 - BN^2) = 0,$$

and conversely, if the above relation be satisfied, the perpendiculars through L, M, N to the sides of the triangle are concurrent



$\because$ $BL^2 + OL^2 = OB^2$ and $CL^2 + OL^2 = OC^2$,
[Bk I, Prop 27

$\therefore$ $BL^2 - CL^2 = OB^2 - OC^2$

Similarly, $CM^2 - AM^2 = OC^2 - OA^2$,

and $AN^2 - BN^2 = OA^2 - OB^2$

Hence by addition we obtain the given relation

To prove the converse theorem, let LO and MO meet in O, then if the perpendicular through N does not pass through O, draw ON' perpendicular to AB, and show that N and N' must be the same point.

EXERCISES

1 Prove this proposition by taking the point O outside the triangle.

2 The right-bisectors of the sides of a triangle are concurrent.

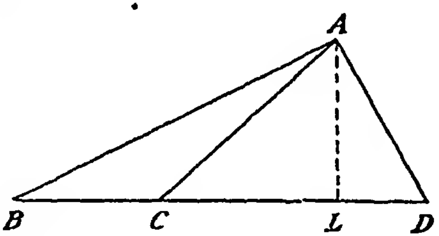
3 The three altitudes of a triangle are concurrent

4 The perpendiculars to the sides of a triangle at the points of contact of the three escribed circles are concurrent

5 If three circles touch externally, the tangents at the points of contact are concurrent

Proposition VII

Triangles which have the same altitude are to one another as their bases



Let AL be the common altitude of $\triangle$s ABC, ACD.

Then $\triangle ABC = \frac{1}{2} AL \cdot BC$,

and $\triangle ACD = \frac{1}{2} AL \cdot CD$ [*Prop* 21, *Cor* 2

Therefore $$\frac{\triangle ABC}{\triangle ACD} = \frac{BC}{CD}$$

Proposition VIII

Ceva's Theorem

If three concurrent lines AO, BO, CO drawn through the angular points of a triangle meet the opposite sides in A', B', C', then

But $$\frac{B'C \cdot C'A \cdot A'B}{BC' \cdot CA' \cdot AB'} = 1 \qquad [Hyp$$

Therefore equating the left-hand sides we get

$$\frac{BA''}{CA''} = \frac{BA'}{CA'}$$

Hence the point A'' coincides with A';

AA', BB', CC' are concurrent

Expressed in words, the condition of concurrence states that—

The product of three alternate segments equals the product of the three remaining segments

Notice also that this condition may be written down thus—

Write the three factors BC, CA, AB above and below a fraction line, accent the first letters above and the second letters below, and equate the result to unity

EXERCISES

1 *The three medians of a triangle are concurrent*

2 *The bisectors of the three angles of a triangle are concurrent*

3 *The three altitudes of a triangle are concurrent*

4. *The straight lines joining the vertices of a triangle to the points of contact of the inscribed circle with the opposite sides, are concurrent*

5 In the figure of this proposition show that—

$$\frac{OA'}{AA'} + \frac{OB'}{BB'} + \frac{OC'}{CC'} = 1$$

6 *The escribed circles of a triangle touch the sides opposite to the angles A, B, C in A', B', C' respectively, prove that AA', BB', CC' are concurrent*

7 With sides 3", 3 2", and 3 7" draw a triangle ABC, divide the sides taken in order at the points A', B', C' in the ratios of 2 3, 3 · 1, and 1 2 respectively

Prove that AA', BB', CC' are concurrent

8 *The sides BC, CA, AB of a triangle taken in order are divided at the points A', B', C', in the ratios of m n, n l, and l m respectively.*

Prove that the lines AA', BB', CC' are concurrent

9 *The lines from the vertices of a triangle to the points of contact of an escribed circle are concurrent*

10 With sides 4″, 4 3″, and 2 7″ draw a triangle, and draw its incircle and three excircles, exhibit the lines of Exs 4, 6, and 9 as concurrent lines

Proposition IX

Menelaus's Theorem

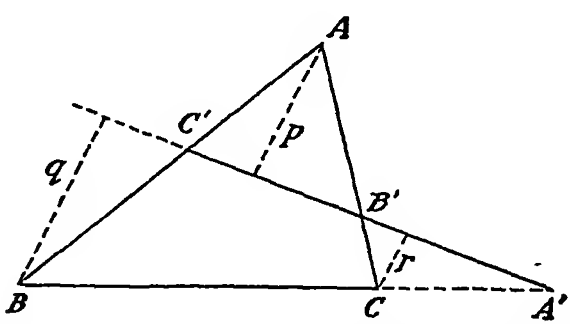
If a transversal meet the sides BC, CA, AB of a triangle in A′, B′, C′, then

$$\frac{B'C}{BC'}\,\frac{C'A}{CA'}\,\frac{A'B}{AB'} = -1;$$

and conversely, if three points A′, B′, C′, taken on the sides BC, CA, AB of a triangle, satisfy the relation

$$\frac{B'C}{BC'}\,\frac{C'A}{CA'}\,\frac{A'B}{AB'} = -1$$

then the points A′, B′, C′ are collinear



From A, B, C draw the perpendiculars p, q, r to the transversal

By similar triangles we have

$$\frac{B'C}{AB'} = \frac{r}{p}, \quad \frac{C'A}{BC'} = \frac{p}{q}, \quad \frac{A'B}{CA'} = -\frac{q}{r}$$

Hence, multiplying, we get

$$\frac{B'C}{BC}\frac{C'A}{CA'}\frac{A'B}{AB'} = -1$$

To prove the converse, let the join of B', C' meet BC in A'' Show that A'' and A' are the same point

Note — The ratio $A'B/CA'$ is negative, for one of the segments in this case must be negative When the transversal cuts all the sides externally, the three ratios must all be negative Hence in all cases the product is negative unity

EXERCISES

1 *The bisectors of the three external angles of a triangle meet the opposite sides in three points, which are collinear*

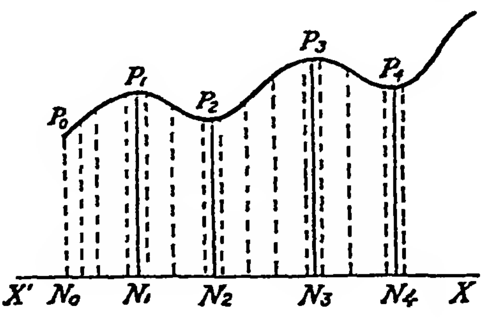
2 *The bisectors of two internal angles of a triangle and the bisector of the third external angle meet the opposite sides in three collinear points*

SECTION II—MAXIMA AND MINIMA

General Method

CONSIDER the perpendicular distances of a point P from a fixed line $X'X$ as the point travels along the curve in the figure

Starting from P_0 the perpendiculars continuously increase until P_1 is reached, and then they begin to decrease.



At P_1 the length of the perpendicular is greater than either its preceding or succeeding lengths.

Beyond P_1 the perpendiculars continuously decrease until P_2 is reached, and then they begin to increase. At P_2 the length of the perpendicular is less than either its preceding or succeeding lengths

After P_2 the perpendiculars increase until P_3, and then

they begin to decrease; and at P_3 the length of the perpendicular is greater than either its preceding or succeeding lengths

Beyond P_3 the perpendiculars decrease until P_4, and then they begin to increase, and at P_4 the length of the perpendicular is less than either its preceding or succeeding lengths

Thus we see that the perpendiculars at P_1 and P_3 are greater than any other perpendiculars in their immediate vicinity, and, on the other hand, the perpendiculars at P_2 and P_4 are less than any other perpendiculars in their immediate vicinity

We say, then, that in the course of change the perpendiculars attain their *greatest* values at P_1 and P_3, and their *least* values at P_2 and P_4

Hence the terms "greatest" and "least," as used here, are relative and not absolute.

Thus P_1N_1 is the greatest perpendicular in the neighbourhood of P_1, though it is less in absolute value than P_3N_3

DEF —*When a geometrical magnitude, which changes continuously according to any law, passes in the course of change through a value greater than either its preceding or succeeding values it is said to be a* maximum, *and, on the other hand, when it passes through a value less than either its preceding or succeeding values it is said to be a* minimum

In the figure given above notice that—

(i) Maxima and minima values occur alternately

(ii) Very near P_1, and on either side of it, there are two perpendiculars of equal length, and if the distance between them be made indefinitely small they will coalesce

into one perpendicular P_1N_1, which is a maximum perpendicular

(iii) Similarly, the minimum perpendicular P_2N_2 is obtained by making two equal perpendiculars, on either side of P_2, coalesce into one

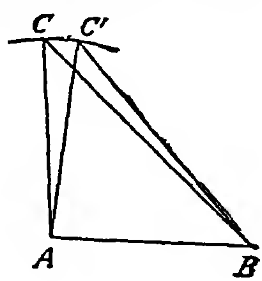
(iv) As a consequence of (ii) and (iii) the tangents to the curve at P_1, P_2, P_3, P_4 are parallel to $X'X$

From the above we conclude that *a maximum or minimum value of a changing magnitude lies between two equal values, and may be obtained by making the two equal values coalesce into one*

In practice a position of maximum or minimum value of a geometrical magnitude is found by assuming that it has the same value for two consecutive positions which are indefinitely near each other

Proposition I

If two sides of a triangle be given in magnitude, the area is a maximum when they contain a right angle.



Let AB, AC be the given sides

Suppose AB is fixed in position, then the locus of C is the circumference of a circle, centre A and radius AC Let ACB be the maximum triangle, and let C' be a point on the circumference indefinitely near to C Then

$$\triangle ACB = \triangle AC'B,$$

$$CC' \parallel AB.$$

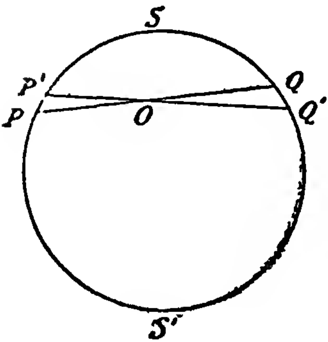
But the join of CC' is the tangent at C;

$\therefore CC' \perp AC$. [Bk. I, Prop 37.

Hence $\angle CAB$ is a rt. $\angle$

Proposition II

Through a given point within a circle draw a chord which shall cut off a segment of minimum area from the circle



Let O be the given point, and let PQ be the minimum chord

Let $P'Q'$ be a consecutive position of PQ

Join PP', QQ'

Then

segment PSQ = segment $P'SQ'$,

$\therefore \triangle OPP' = \triangle OQQ'$

But these are similar triangles, [Bk I, Prop 40

$OP^2 = OQ'^2$, [Bk I, Prop 50

i e ultimately $OP = OQ$

Hence the required chord is bisected at the point

Cor 1 —*The same chord cuts off the maximum segment $PS'Q$*

Cor. 2 —*The same chord is the chord of minimum length through the given point*

Cor 3 —*The straight line drawn through a given point, which cuts off the triangle of minimum area from two intersecting straight lines, is bisected at that point*

Proposition III

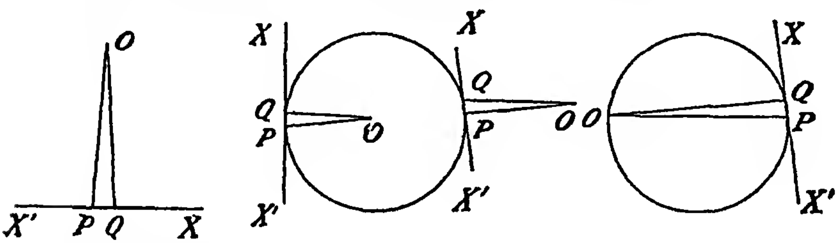
Of straight lines drawn from a given point to a given straight line, the minimum is that which is perpendicular to the given straight line

Proposition IV

The maximum or minimum line, which can be drawn from a given point to a given circle, lies in the direction of the diameter through the given point.

Proposition V

The diameter is the maximum chord of a circle



Let OP be a maximum or minimum, and let OQ be a consecutive position of OP, so that PQ is indefinitely small.

Then $$OP = OQ,$$
$$\angle OPQ = OQP = \text{rt. } \angle,$$
when P and Q coincide,
$$OP \perp X'X$$

In the second and third figures $X'X$ is a tangent and PO a line through the point of contact perpendicular to the tangent,

$\therefore$ OP passes through the centre

Proposition VI

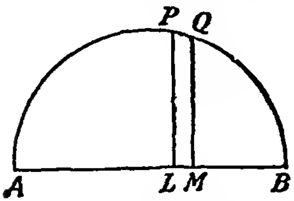
If a given straight line be divided into two parts the rectangle contained by them is a maximum when they are equal

Let AB be the given line and L the point of division.

On AB describe a semicircle, and let LP perpendicular to AB meet the circumference in P

Then $$AL \cdot LB = PL^2.$$

Suppose PL is a maximum and QM a consecutive position of PL, so that PQ is indefinitely small



Then $$PL = QM,$$
$$PQ \parallel LM$$

Therefore the tangent at P is parallel to AB

Hence P is the middle point of the arc AB

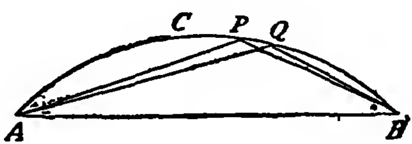
Therefore L is the middle point of AB

COR 1—*If the sum of two straight lines be given, the rectangle contained by them is a maximum when they are equal*

COR 2—*Of all isoperimetrical rectangles the square has the maximum area*

PROPOSITION VII

In a given segment of a circle to inscribe a triangle of maximum area.



Let ACB be the given segment.

Suppose APB is the maximum triangle and Q a point very near P.

Then $$\triangle APB = \triangle AQB,$$
$$\therefore PQ \parallel AB$$

Hence the tangent at P is parallel to AB

Therefore P is the middle point of the arc ACB

Thus when the area APB is a maximum the triangle APB is isosceles

COR. 1.—*Of all polygons of a given number of sides in-*

scribed in a circle that which is regular has a maximum area

For suppose n-2 of the sides are kept fixed, and the remaining two AP and BP are allowed to vary, then the area of the polygon is a maximum when $AP = BP$

Similarly, by making PB and the next consecutive side, say BD, to vary, we see that the area is a maximum when $PB = BD$

Thus, when all the sides vary, the area is a maximum when the polygon is equilateral

But an inscribed equilateral polygon is also equiangular

COR 2 —*Given the base and vertical angle of a triangle, its area is a maximum when the triangle is isosceles.*

Algebraical Methods

Besides the general method illustrated in Props I -VII, we may employ Algebra for the determination of the maxima and minima values of geometrical magnitudes

PROPOSITION VIII

If a straight line be divided into two parts the sum of the squares on them is a minimum when they are equal.

Let x and y be the two parts

Consider the identity

$$2(x^2+y^2) = (x+y)^2 + (x-y)^2$$

Both the terms on the right-hand side are positive, and the first term is constant.

Hence $2(x^2+y^2)$ is a minimum when $(x-y)^2$ vanishes.

Therefore x^2+y^2 is a minimum when $x=y$

In the same way, by employing the identity

$$4ab = (a+b)^2 - (a-b)^2,$$

we may prove that—

If a straight line be divided into two parts, the rectangle contained by them is a maximum when they are equal

Again, since $(a+b)^2 = 4ab + (a-b)^2$, we have—

If the area of a rectangle be given, its perimeter is least when it is a square

It also follows from the above that—

If the perimeter of a rectangle is given, its area is greatest when it is a square

Proposition IX

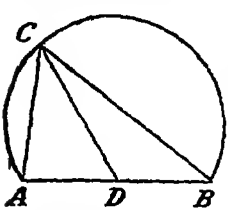
If the rectangle contained by two straight lines is given, the sum of the squares on them is a minimum when they are equal

This proposition depends on the identity

$$a^2 + b^2 = 2ab + (a-b)^2$$

Proposition X

Given the base and vertical angle of a triangle, the sum of the squares of its sides is a maximum when it is isosceles



Let AB be the given base, then the locus of the vertex C is the segment of a circle described on AB as base and capable of the given angle.

If D be the middle point of AB, then

$$BC^2 + AC^2 = 2AD^2 + 2CD^2.$$

Now AD is constant, therefore the left-hand side is a maximum when CD is a maximum.

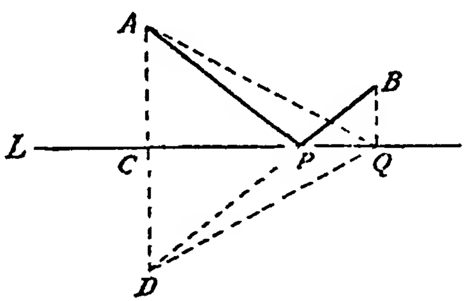
By the general method it can easily be shown that CD is a maximum when the tangent at C is parallel to AB.

Geometrical Methods

We shall now give a few examples of geometrical methods for finding maxima and minima values

PROPOSITION XI

If from two given points A and B on the same side of a given line L straight lines AP, BP be drawn to a point P in L, their sum is a minimum when they make equal angles with L



Draw $AC \perp L$, and produce it to D, making CD equal to CA.

Join DB, cutting L in the point P.

Then prove that—

(i) AP, PB make equal angles with L

(ii) $AP + PB = BD$.

(iii) If Q be any other point on L, then

$$AQ + QB = QD + QB.$$

(iv) From (ii) and (iii) deduce that

$$AQ + QB > AP + PB.$$

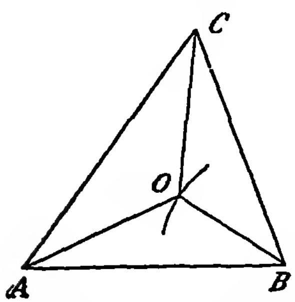
Hence $AP+PB$ is a minimum when AP, PB make equal angles with L

COR 1 —*When A and B are on opposite sides of L the difference of AP and BP is a maximum when they make equal angles with L*

COR 2 —*If from two given points A and B on the convex side of the arc of a circle straight lines AP, BP be drawn to a point P on the arc, their sum is a minimum when they make equal angles with the tangent to the circle at P*

For the arc of the circle in the neighbourhood of P may be replaced by a straight line, viz the tangent at P

COR. 3 —*Within an acute-angled triangle find a point the sum of whose distances from the angular points of the triangle is a minimum*



Let O be the required point

Suppose one of the distances BO is kept constant and the other two allowed to vary

Then the locus of O is a circle, centre B and radius BO

We have to find a point O on this circle such that $OA+OC$ is a minimum

By Cor 2, OA, OC must make equal angles with the tangent at O, hence they must also make equal angles with the radius OB

Therefore $\angle BOC = \angle BOA$

Similarly, by keeping OA constant we can show that

$$\angle BOA = \angle COA$$

Hence $\angle BOC = \angle COA = \angle BOA = 120°$

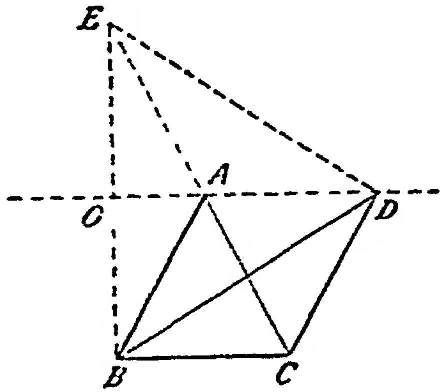
The above analysis leads to the following construction —

On AB, BC describe segments, each capable of $120°$,

then the point of intersection of these segments is the required point.

Proposition XII

Of all equal triangles on the same base the isosceles triangle has the least perimeter.



Let ABC be an isosceles triangle and DBC any other triangle on the same base and between the same parallels.

Draw $BO \perp AD$ and produce it, cutting off OE equal to OB.

Then proceeding as in Prop XI. prove that

$$BD + CD > AB + AC$$

COR 1 —*Of all n gons which have the same area the perimeter of an equilateral n-gon is a minimum.*

For keeping n-2 sides fixed the remaining two sides must be equal Thus the figure must be equilateral.

COR. 2 —*Of all triangles of equal area the equilateral triangle has a minimum perimeter*

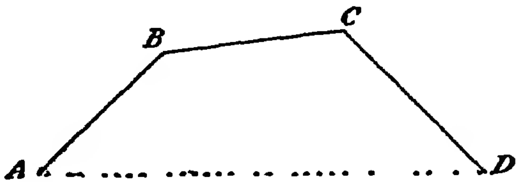
COR 3 —*Of all parallelograms of equal area the rhombus has a minimum perimeter.*

Proposition XIII

Being given three sides AB, BC, CD of a quadrilateral $ABCD$. its area will be a maximum when A, B, C, D lie on a circle of which the fourth side AD is a diameter.

Suppose ABD is not a rt. $\angle$; then keeping BCD fixed,

the area $ABCD$ can be made greater by putting AB, BD at right angles. [*Prop* I.



Hence ABD is a right angle

Similarly, ACD is a right angle

Therefore the circle on AD as diameter passes through B, C

COR 1—*If n-1 sides of an n-gon be given in order and magnitude, its area is a maximum when its angular points lie on a circle of which the nth side is a diameter*

COR. 2—*When the extremities of a curved line are connected by a straight line the area enclosed is a maximum when the form is a semicircle*

For the curved line may be regarded as the ultimate form of a very large number of short straight lines placed end to end The result then follows from Cor 1

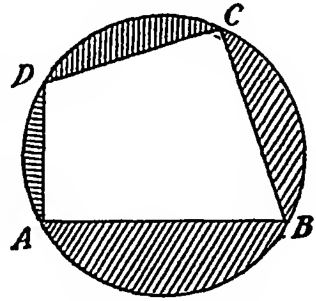
COR. 3—*When the perimeter of a closed figure is given, its area is a maximum when its form is a circle*

Consider the circle enclosed by the given perimeter and draw a diameter Then the area of either semicircle would be diminished by altering the circular form [*Cor.* 2.

PROPOSITION XIV

When the sides of a quadrilateral are given in order and magnitude, the area is a maximum when it can be inscribed in a circle

Suppose *ABCD* inscribed in a circle, and imagine the sides of the quadrilateral movable about hinges at the angular points, and having the shaded circular segments permanently attached to them. If now the quadrilateral be deformed in any manner, the external perimeter of the figure formed by the four circular arcs will remain unchanged, and since its form is no longer circular its area is diminished [*Prop* XIII, *Cor* 3]



But the only area that has undergone change in this operation is that of the quadrilateral.

Hence any deformation which destroys the cyclic character of the quadrilateral makes its area less

Thus a cyclic quadrilateral is greater than any other quadrilateral having corresponding sides of the same length

COR —*When the sides of a polygon are given in order and magnitude, the area is a maximum when the angular points are concyclic*

EXERCISES ON MAXIMA AND MINIMA

1 If a given straight line be divided into any number of parts, the sum of their squares is a minimum when they are equal

2 If a given straight line be divided into any number of parts, their continued product is a maximum when they are equal

3 Find the maximum and minimum distances between two non-intersecting circles

4. *P* is a fixed point between two given lines *OX*, *OY*, through *P* draw a line *APB* terminated by *OX*, *OY* such that the rectangle *PA* *PB* shall be a minimum.

5 In a semicircle inscribe the maximum triangle.

6 In a square inscribe the minimum square

7 If the sides of a parallelogram be given, the area is a maximum when the parallelogram is a rectangle

8 If the diagonals of a parallelogram be given, its area is a maximum when it is a rhombus

9 If the diagonals of a quadrilateral be given, the area is a maximum when they are at right angles

10 A and B are two given points on the convex side of a given circle, find a point P on the circumference such that PA^2+PB^2 may be a minimum

11 Find a point within a square such that the sum of the squares of its distances from the sides is a minimum

12 One circle lies entirely within another, find the maximum and minimum chords of the outer which touch the inner

13 In a given circle inscribe the maximum triangle

14. If two tangents be drawn to a circle, find the points of contact of a third tangent which cuts off a maximum or minimum triangle

15 If two tangents be drawn to a circle, find the point of contact of the minimum tangent intercepted between them

16 Of all n gons circumscribing a circle, that which is regular has a minimum area

17 Of all n-gons circumscribing a circle, that which is regular has a minimum perimeter

18 Through one of the points of intersection of two given circles draw the maximum straight line terminated by the circumferences

19 Given the base and area of a triangle, prove that the vertical angle is a maximum when the triangle is isosceles

20 From a point P in the side BC of a triangle ABC, PQ, PR are drawn parallel to the sides BA, CA; find when the parallelogram $PQAR$ is a maximum.

SECTION III—THE TRIANGLE.

In discussing the properties of the triangle ABC we shall use the following notation throughout this section, without defining again the letters employed —

(i.) D, E, F are the middle points of the sides opposite to the angles A, B, C.

(ii.) L, M, N are the feet of the perpendiculars from the angles A, B, C to the opposite sides.

(iii.) D', E', F' are the points where the internal bisectors of the angles A, B, C meet the opposite sides; and D'', E'', F'' the corresponding points for the external bisectors

(iv.) The points of concurrence of the medians and altitudes are G and P respectively.

(v.) O circumcentre, I incentre, and I_1, I_2, I_3 excentres; R circumradius, r inradius, and r_1,r_2,r_3 exradii

(vi.) μ the mid point of PO, and α, β, γ the mid points of PA PB, PC.

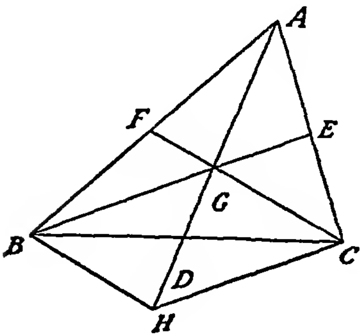
Proposition I

The Medians

The medians of a triangle are concurrent, and their point of concurrence is a point of trisection of each median

Draw the medians through B and C

We shall prove that AG produced bisects BC.



Draw $BH \parallel FC$, meeting AG produced in H

Join CH

In $\triangle ABH$, FG is drawn through the mid point of AB parallel to BH

Therefore G bisects AH

In $\triangle ACH$, GE joins the mid points of AH and AC, therefore it is parallel to CH

Hence $BGCH$ is a $\parallel^m$

Therefore the diagonals GH and BC bisect one another at D.

We have proved that $GA = GH$, and $GH = 2GD$

Therefore $GA = 2GD$

Hence G is a point of trisection of AD

The point G is called the **centroid** of the triangle ABC In mechanics it is called the **centre of gravity.**

EXERCISES

1 Prove that the sum of the medians of a triangle is greater than three fourths of the sum of its sides

2 If FE meet AD in J, prove that $2AJ = AD = 6JG$

3 Through D draw DM parallel to BE, meeting FE produced in M, and join AM, prove that the triangle MAD has its three sides equal to the medians of the triangle ABC

4 *Given the three medians of a triangle, construct it.*

5 The medians of a triangle are 1 7″, 2″, and 2 2″, construct it.

6 The medians of a triangle are 3 cm, 4 cm., and 4 5 cm; construct it.

7 Prove that

$$AG^2+BG^2+CG^2=\frac{1}{3}(a^2+b^2+c^2)$$

8 A and B are two fixed points without a circle whose centre is C; find a point G on the circumference such that AG^2+BG^2 may be a minimum

If F be the middle point of AB, show that CGF is a straight line

9 Apply the last result to find a point within a triangle, the sum of the squares of whose distances from the angular points is a minimum

10 Show that the distance between the centroids of the triangles ABD, ACD is $\frac{1}{3}BC$

11 The base and area of a triangle being given, find the locus of its centroid

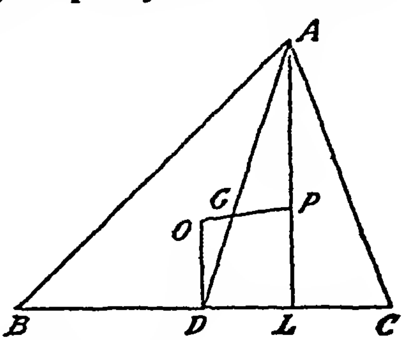
12 The base and vertical angle of a triangle being given, show that the locus of the centroid is an arc of a circle

13 The base of a triangle is fixed, and the vertex moves on a fixed straight line; prove that the locus of the centroid is another fixed straight line.

Proposition II

The Perpendiculars

The perpendiculars from the angular points of a triangle to the opposite sides meet in a point called the **orthocentre**; *and the distance of each angular point from the orthocentre is twice the distance of the circumcentre from the side opposite to that angular point*



Join OG, and produce it to meet AL in P.

Then P is the orthocentre

$\because OD \parallel AL$,

$\therefore \triangle$s ODG, PAG are similar,

and $\because AG=2GD$, [Prop. I.

$\therefore PA=2OD$ and $PG=2OG$

Similarly, if OG produced meet another perpendicular in the point P', then

$$P'G = 2OG$$

Therefore P' is coincident with P

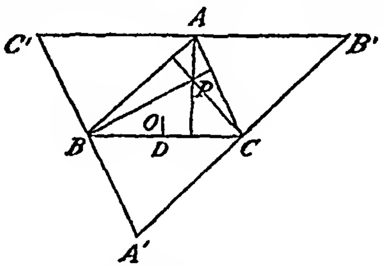
Hence all the perpendiculars pass through the same point P, and

$$PA, PB, PC = 2OD, 2OE, 2OF$$

DEF —*The triangle formed by joining the feet of the perpendiculars is called the* **pedal** *or* **orthocentric** *triangle*

EXERCISES

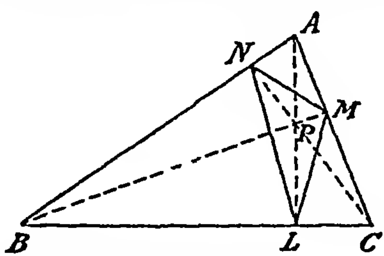
1. Through A, B, C draw parallels to the opposite sides forming



the triangle $A'B'C'$ Prove that the circumcentre of $A'B'C'$ is the orthocentre of ABC, and that $PA = 2OD$

2 The circumcircles of the triangles ABC, PBC, PCA, PAB are equal.

3 The quadrilaterals $PMAN$, $PNBL$, $PLCM$ are cyclic



4. The angles of the pedal triangle are

$$180° - 2A,\ 180° - 2B,\ 180° - 2C$$

5 *The sides of the pedal triangle make equal angles with the sides of the original triangle*

6 *The pedal triangle is the minimum triangle which can be inscribed in the original triangle*

7. *The orthocentre of the original triangle is the incentre of the pedal triangle*

8 The angular points of the original triangle are the excentres of the pedal triangle

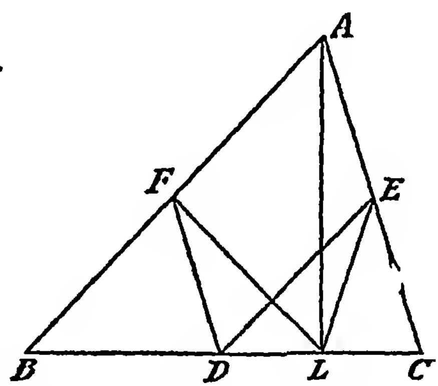
9 The base BC and the opposite angle A being given, show that PA is of constant length

10 The lines joining the circumcentre and orthocentre to any angular point of a triangle are equally inclined to the sides which meet in that angular point

Proposition III

The Nine-Point Circle

In any triangle the three middle points of the sides, the three feet of the perpendiculars, and the three middle points of the joins of the orthocentre and vertices all lie on a circle, called the **nine-point circle**, *whose radius is half the circumradius of the triangle, and whose centre is the middle point of the join of the circumcentre and orthocentre*



We shall first prove that the circle through D, E, F passes through L, M, N

ALC is a right-angled triangle and E is the mid point of its hypotenuse,

$$\therefore \angle ELA = \angle EAL.$$

Similarly, $\angle FLA = \angle FAL$;

$$\therefore \quad \angle ELF = \angle EAF$$

But $AEDF$ is a $\|^m$,

$$\angle EAF = \angle EDF$$

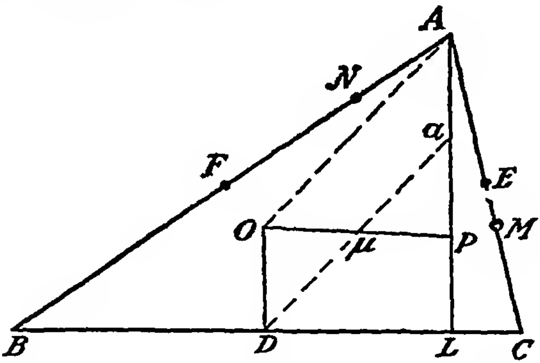
Therefore $\angle ELF = \angle EDF$

Hence the circle through D, E, F will pass through L. [Bk I, Prop 40

Similarly, it will pass through M and N

We shall next prove that this circle passes through the mid points of PA, PB, PC

Since DL is a chord of this circle, its centre must lie on the right-bisector of DL.



But the right-bisector of DL passes through the mid point of OP

Similarly, the centre must lie on the right-bisectors of ME and FN, and these lines also pass through the mid point of OP

Hence μ, the mid point of OP, is the centre of this circle.

Join $D\mu$, and produce it to meet AP in a.

Then from similar triangles

$$OD = aP, \text{ and } \mu D = \mu a,$$

but $OD = \frac{1}{2}AP$,

$$aP = \frac{1}{2}AP$$

Hence α is the mid point of AP.

Therefore the circle whose centre is μ and radius μD will pass through the mid point of AP

∴ Similarly, it will pass through β, γ, the mid points of BP and CP

Hence the circle whose centre is μ passes through the nine points $D, E, F, L, M, N, \alpha, \beta, \gamma$

Again, ∵ $AOD\alpha$ is a $\|^{m}$,

$$\therefore D\alpha = OA = R,$$

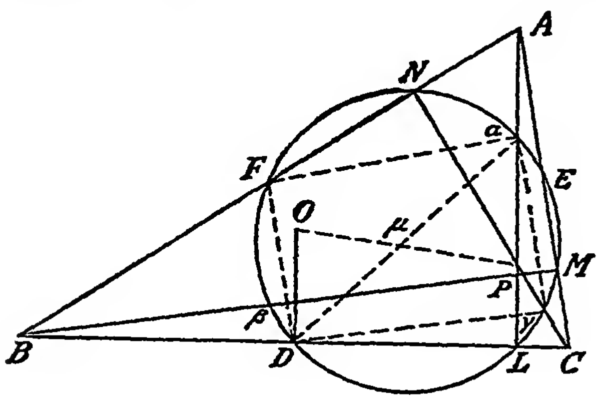
$$\therefore \mu D = \tfrac{1}{2}R$$

Note.—The nine-point circle is sometimes called the **medioscribed circle** of the triangle, and its centre, μ, the **midcentre** of the triangle

EXERCISES

1 Prove that the triangles ABC, PBC, PCA, PAB have the same orthocentre

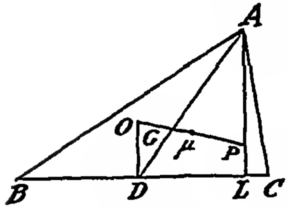
2 Prove that the triangles ABC, PBC, PCA, PAB have the same nine-point circle



3 Prove that $D\alpha = E\beta = F\gamma$

4 Prove that $EF\beta\gamma$, $DE\alpha\beta$, $FD\gamma\alpha$ are rectangles whose diagonals intersect in μ.

5 Prove that $FD\gamma\alpha$, $DE\alpha\beta$, $EF\beta\gamma$ are rectangles



6 *Prove that the circumcentre, the centre of gravity, the midcentre, and the orthocentre of a triangle are collinear*

7 Prove that $OG : G\mu : \mu P = 2 : 1 : 3$

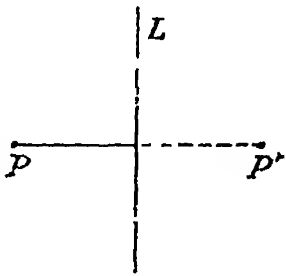
8 The nine point circle of an equilateral triangle is its incircle

9 Construct a triangle, having given the circumcircle, the orthocentre, and one angular point

10 The six joins of the incentre and three excentres are bisected by the circumcircle

DEF —*If from any point P a perpendicular be drawn to a line L, and produced through the line to a point P′ at the same distance from L as P, then P′ is called the* reflexion, *or* image, *of P in the line L*

Suppose L to be a looking-glass, then the reason for giving this name will be easily discovered



Proposition IV

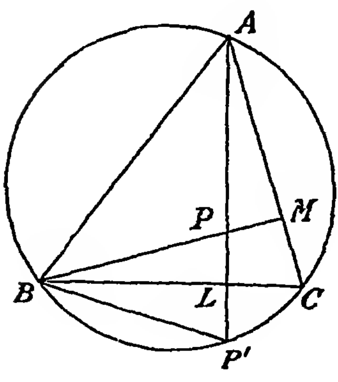


Image of the Orthocentre

The image of the orthocentre in any side of a triangle lies on the circumcircle

From similar Δs,

$$\angle CAL = \angle CBM$$

Also

$$\angle CAL = \angle CBP'$$

[Bk I, Prop 40.

Therefore $\angle PBL = \angle P'BL$

Hence the right-angled triangles PBL, $P'BL$ are congruent, and

$$PL = P'L$$

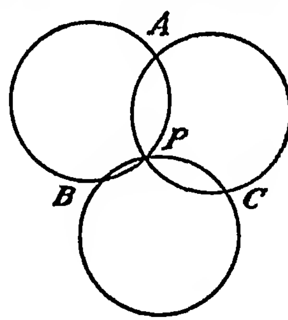
EXERCISES

1 Defining the image of a plane figure to be the locus of the images of its several points, prove that the image of a straight line is a straight line, and the image of a circle is a circle

2 Prove that the angle between a straight line and its reflexion is bisected by the axis of reflexion

3 Given the base and vertical angle of a triangle, find the locus of the orthocentre

4. Two equal circles pass through two fixed points, show that each is the locus of the orthocentres of triangles inscribed in the other on the join of the fixed points as base



5 Three equal circles pass through the point P and intersect one another again in the points A, B, C Prove that P is the orthocentre of the triangle ABC

PROPOSITION V

The Simson Line

The feet of the perpendiculars, from any point on the circumcircle of a triangle, to the sides of the triangle, lie on a straight line called the Simson Line

Let X, Y, Z be the feet of the perpendiculars from Q to the sides of the triangle

Join YZ, ZX, AQ, BQ

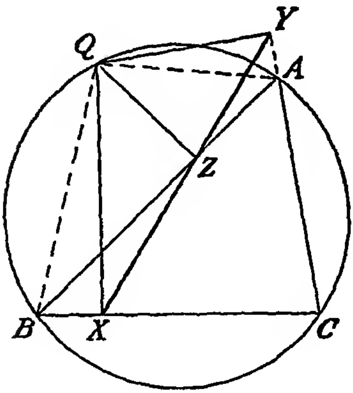
Then $\angle AZY = \angle AQY$ [*AYQZ is cyclic*

$= 90° - \angle QAY$

$= 90° - \angle QBX$ [*Prop* 42, *Cor* 1.

$$= \angle BQX$$

$$= \angle BZX \qquad [BXZQ \text{ is cyclic}$$



Therefore X, Z, Y are collinear

Note —The line XZY, on which the feet of the perpendiculars lie, is sometimes called the **pedal line** of Q with respect to the triangle

EXERCISES

1 *If from any point Q the perpendiculars OX, OY, OZ be drawn to the sides BC, CA, AB of a triangle, and if the three points X, Y, Z be collinear, the point O must lie on the circumcircle of ABC*

Prove that $\angle QBX = \angle QAY$.

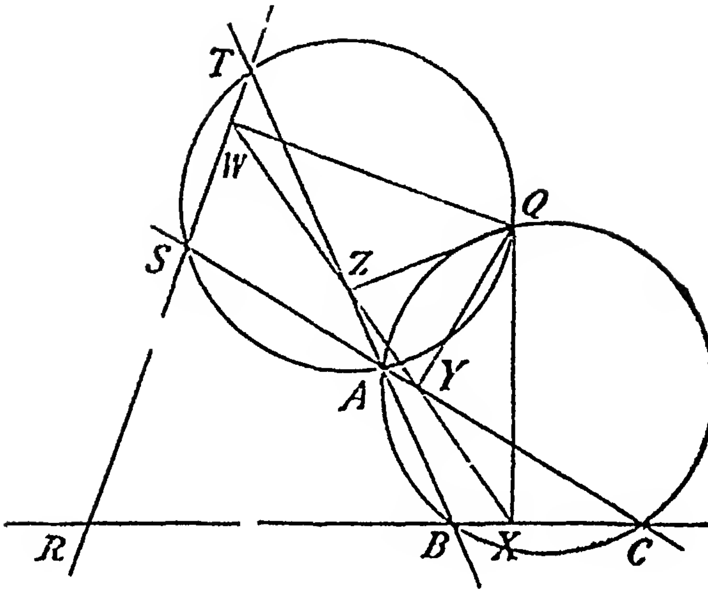
Hence $ACBQ$ is cyclic [*Bk I*, *Prop* 42, *Cor* 2

2 *The four circumcircles of the four triangles formed by four straight lines, no two of which are parallel, have a common point of intersection*

Draw the circumcircles of two of the Δs formed by the lines These circles intersect at A, and let their second point of intersection be Q From Q draw perpendiculars QX, QY, QZ, and QW to the sides of the two Δs

Q is on the circumcircle of ABC,

X, Y, Z are collinear



And $\qquad Q$ is on the circumcircle of AST;

$\therefore Y, Z, W$ are collinear,

$\therefore X, Y, Z, W$ are collinear

Again, $\qquad \because X, Y, W$ are collinear,

Q is on the circumcircle of ΔCRS

And $\qquad X, Z, W$ are collinear,

Q is on the circumcircle of ΔBRT. [Ex 2.

Thus Q lies on the circumcircles of the Δs ABC, AST, CRS, and BRT

3 *The pedal line of any point bisects the join of the point and the orthocentre*

Let AL produced meet the circumcircle in P', and let AP' cut the pedal line in S and BC in T

Also let QP meet the pedal line in V

Prove that S is the mid point of QT, and prove that $P'T$ is parallel to the pedal line The result follows

Proposition VI

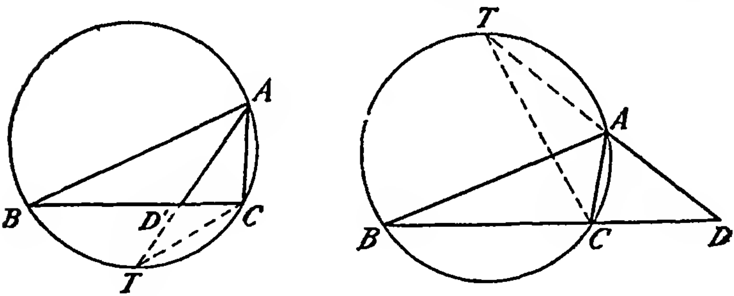
The Bisector of an Angle

If an angle of a triangle be bisected by a straight line which meets the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by the segments of the base, together with the square on the bisector of the angle.

Draw the circumcircle of the triangle, and produce the bisector AD' to meet it in T

Join CT

Then $\angle BAD' = \angle CAD'$, [*Hyp*



and $\angle ABD' = \angle ATC$, [*Bk I, Prop* 40.

Δs ABD', ATC are similar,

$$\frac{BA}{AD'} = \frac{AT}{AC}$$

Hence
$$\begin{aligned} BA\ AC &= AD'\ AT \\ &= AD'\ D'T + AD'^2 \\ &= BD'\ CD' + AD'^2 \end{aligned}$$

[*Bk I, Prop* 44

Cor —*If AD'' be the external bisector of the angle A, then*

$$BA\ AC = BD''\ CD'' - AD''^2$$

With the same argument we have from the second figure

$$\begin{aligned} BA \cdot AC &= AD'' \cdot AT \\ &= AD'' \ (D''T - AD'') \\ &= AD'' \ D''T - AD''^2 \\ &= BD'' \ CD'' - AD''^2. \end{aligned}$$

[*Bk I, Prop.* 44.

EXERCISES

1 If f be the length of the bisector of A, prove that

$$f^2 = \frac{4bc}{(b+c)^2} s(s-a)$$

2 If f_1 be the bisector of the exterior angle at A, prove that

$$f_1^2 = \frac{4bc}{(c-b)^2}(s-b)(s-c)$$

3 Prove that in both figures T bisects the arc BTC

4 Prove that $TD \ TA = TC^2$

5 Prove that $TD'' \ TA = TC^2$

6 Prove that $AB \ CT = AT \ BD'$.

7 Prove that $AB \ CT = AT \ BD''$

PROPOSITION VII

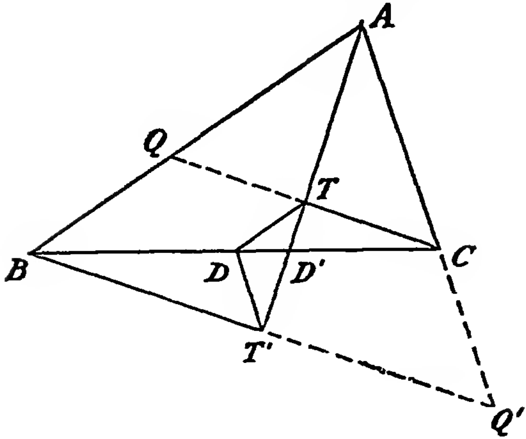
If from the extremities of the base of a triangle perpendiculars be drawn on the internal bisector of the vertical angle, the join of the middle point of the base and the foot of either perpendicular is equal to half the difference of the sides of the triangle

Let BT', CT be the perpendiculars, then

$$DT = \tfrac{1}{2}(AB - AC) = DT'.$$

Cut off AQ, $AQ' = AC$, AB.

Prove that CTQ, $BT'Q'$ are straight lines, and that T, T' are their mid points

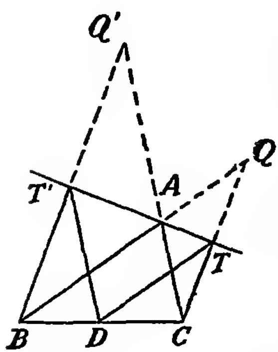


Hence $DT, DT' \parallel BQ, CQ'$.

Also $BQ = AB - AC = CQ'$.

Therefore $DT = \frac{1}{2}(AB - AC) = DT'$.

EXERCISES



1 *If from the extremities of the base of a triangle perpendiculars be drawn on the external bisector of the vertical angle, the join of the middle point of the base and the foot of either perpendicular is equal to half the sum of the sides of the triangle*

2 If DT, DT', produced if necessary, meet the sides of the triangle in K, R, then $AKDR$ is a parallelogram

3 Prove that $\angle BCQ = \frac{1}{2}(\angle C \sim \angle B)$

4 Prove that $\angle ACQ = \frac{1}{2}(\angle C + \angle B)$

5 Prove that $\angle BD'Q = \angle C \sim \angle B$

6 Given the base of a triangle and the difference of the other two sides, find the locus of the feet of the perpendiculars from the extremities of the base on the bisector of the internal vertical angle

7. Given the base of a triangle and the sum of the other two sides, find the locus of the feet of the perpendiculars from the extremities of the base on the bisector of the external vertical angle.

8. Prove that $\angle LAD' = \frac{1}{2}(\angle C - \angle B)$

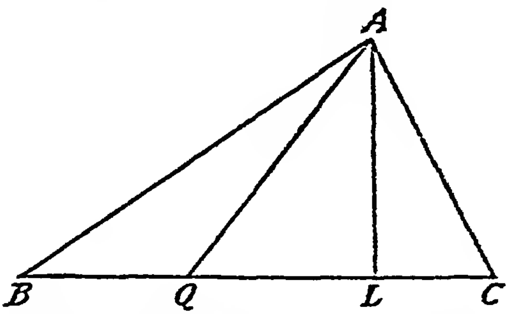
Proposition VIII

If the base BC of the triangle ABC be divided in Q, so that

$$mBQ = nCQ,$$

then

$$mAB^2 + nAC^2 = mBQ^2 + nCQ^2 + (m+n)AQ^2.$$



For $\qquad mAB^2 = m(BQ^2 + AQ^2 - 2BQ \cdot QL)$

and $\qquad nAC^2 = n(CQ^2 + AQ^2 + 2CQ \cdot QL)$

[*Bk. I, Props.* 28, 29

Taking account of the given relation, the required result follows on addition.

EXERCISES

1. *If the base BC of a triangle ABC be divided externally in Q, so that*

$$mBQ = nCQ,$$

then $\qquad mAB^2 - nAC^2 = mBQ^2 - nCQ^2 + (m-n)AQ^2.$

2. Show that the theorem

$$AB^2 + AC^2 = 2AD^2 + 2BD^2$$

is a particular case of the general theorem proved above.

3 The sides opposite the angles A, B, C being 3 5″, 2 5″, and 3″, draw the triangle ABC, and divide the base BC in Q, so that

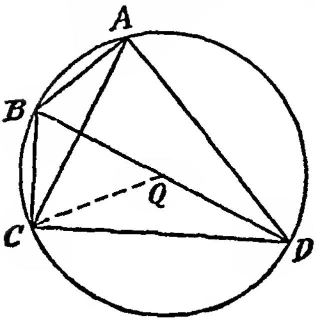
$$BQ \quad QC = 3 \quad 4$$

Calculate the length of AQ

Proposition IX

Ptolemy's Theorem

The rectangle contained by the diagonals of a cyclic quadrilateral is equal to the sum of the rectangles contained by its opposite sides



Make $\angle DCQ = \angle ACB$,

then $\angle BCQ = \angle ACD$

Also $\angle CBQ = \angle CAD$

[*Bk I, Prop* 40

Therefore $\triangle$s BCQ, ACD are similar,

hence $$\frac{AD}{AC} = \frac{BQ}{BC}$$

Therefore $AD \quad BC = BQ \quad AC$

Similarly, $AB \quad CD = DQ \quad AC$

Hence by addition we get

$$AD \quad BC + AB \quad CD = BD \, . \, AC.$$

EXERCISES

1 If a point be taken anywhere on the circumcircle of an equilateral triangle, its distance from one of the angular points is equal to the sum of its distances from the remaining angular points

2 Prove that $BC \quad CD = AC \quad CQ$

3 If the diagonals of a cyclic quadrilateral be at right angles, the

sum of the rectangles contained by opposite sides is equal to twice the area of the quadrilateral.

4. If the angle between the diagonals of a cyclic quadrilateral be half the angle of an equilateral triangle, the sum of the rectangles contained by its opposite sides is four times the area of the quadrilateral.

5 In the figure of this proposition if AC, BD intersect in R, prove that

$$\frac{AB\cdot BC}{AD\cdot DC}=\frac{BR}{DR}.$$

SECTION IV—THE CIRCLE

Orthogonal Circles

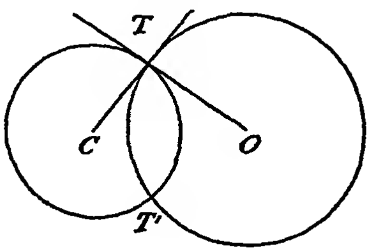
DEF —*The* **angle of intersection** *of two circles is the angle between their tangents at the point of intersection*

DEF —*Two circles are said to cut* **orthogonally** *when their angle of intersection is a right angle*

PROPOSITION I

To describe a circle having a given centre and cutting a given circle orthogonally

Let O be the given centre and C the centre of the given circle



Draw the tangent OT to the circle whose centre is C.
Join CT.

Then $CT \perp OT$, [*Bk I, Prop.* 37.

. CT touches the circle O.

Thus the tangents to the two circles at their point of intersection are at right angles Hence the circles cut orthogonally.

Note.—In the immediate neighbourhood of the point T each circle coincides with its tangent at T Hence the circles are said to be at right angles at T when their tangents are at right angles at T

EXERCISES

1 Show that the two circles cut orthogonally at T'

2 *When two circles cut orthogonally, the sum of the squares of their radii is equal to the square of the distance between their centres*

3 If the sum of the squares of the radii of two circles be equal to the square of the distance between their centres the two circles cut orthogonally

4 Draw two circles with radii 25 mm and 35 mm cutting one another orthogonally

5 Draw two equal circles of one inch radius cutting one another orthogonally, and find the length of their common tangent

The Radical Axis

Proposition II

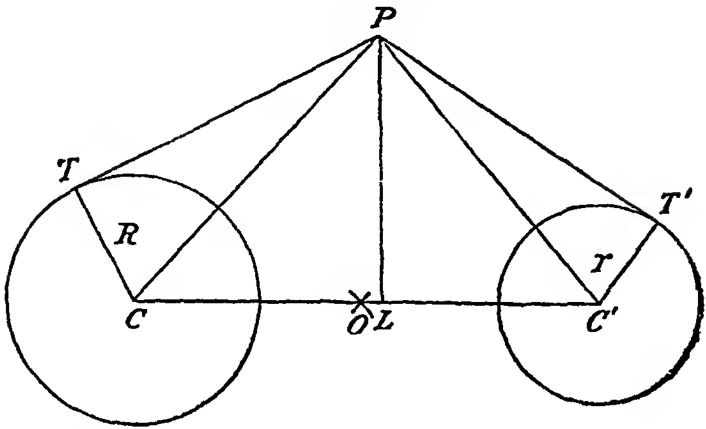
The locus of points from which the tangents drawn to two given circles are equal is a straight line perpendicular to the line of centres

This straight line is called the **radical axis** of the two circles

Let P be a point on the locus, then $PT = PT'$.

Draw $PL \perp CC'$, and let O be the middle point of CC'.

Now $$PT^2 = CP^2 - R^2$$
$$= PL^2 + CL^2 - R^2,$$
and $$PT'^2 = C'P^2 - r^2$$
$$= PL^2 + C'L^2 - r^2.$$



Hence $$CL^2 - R^2 = C'L^2 - r^2,$$
$$\therefore CL^2 - C'L^2 = R^2 - r^2,$$
$$\therefore (CL - C'L)(CL + C'L) = R^2 - r^2;$$
$$2OL \cdot CC' = R^2 - r^2$$

Since CC' and $R^2 - r^2$ are given, L is a fixed point; and P lies on the perpendicular to the line of centres through L

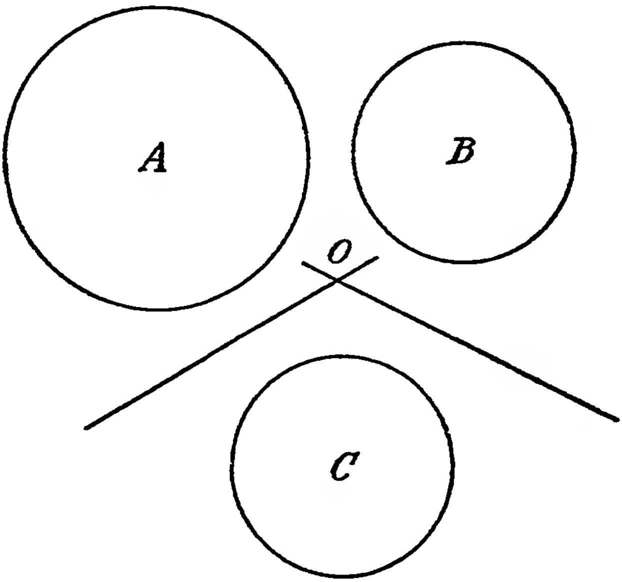
Thus the radical axis is perpendicular to the line of centres, and its position, with respect to the middle point of the line of centres, is determined by the equation last written

EXERCISES

1 *The radical axis of two intersecting circles is their common chord produced both ways*

For if from any point on the common chord, external to the circles, tangents be drawn to the circles, they are equal (See the Exercises to Bk I, Prop 44.)

2 *The radical axes of three circles taken in pairs are concurrent*



Let the radical axis of B and C meet the radical axis of C and A in the point O

$\because$ O is on the radical axis of B and C,

$\therefore$ tangent from O to C=tangent from O to B,

and $\because$ O is on the radical axis of C and A,

$\therefore$ tangent from O to C=tangent from O to A,

tangent from O to B=tangent from O to A

Hence O is on the radical axis of A and B

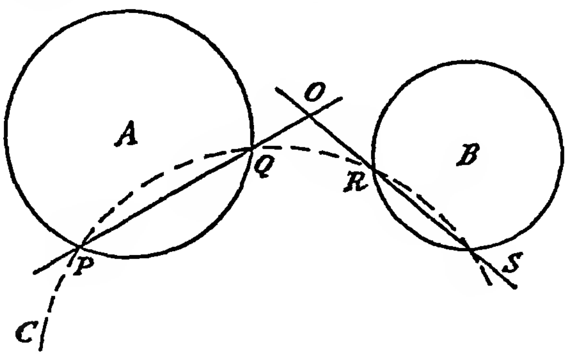
DEF.—*The point of concurrence of the radical axes of three circles is called the* radical centre *of the circles*

3 *Any circle whose centre is on the radical axis of two given circles, and which cuts one of them orthogonally, will also cut the other orthogonally.*

For the radius of the orthogonal circle must be equal in length to the tangent from its centre. [*Prop. I.*

4 *A circle whose centre is the radical centre of three given circles, and which cuts one of them orthogonally, cuts all three orthogonally*

5 *Draw the radical axis of two given circles*



Draw any circle C which cuts both the circles A and B in the points P, Q, R, S

Let PQ, SR meet in O

The chord PQO is the radical axis of C and A, and the chord SRO is the radical axis of C and B

Hence O is the radical centre of A, B, C But the radical centre is a point on the radical axis of A and B

Thus we have found one point O on the radical axis of A and B

Similarly, by drawing any other circle which cuts both A and B we can determine another point O' on the radical axis of A and B

Then OO' is the required radical axis Or draw a perpendicular from O on the line of centres

6 Find the radical centre of three given circles

7 Draw a circle to cut three given circles orthogonally

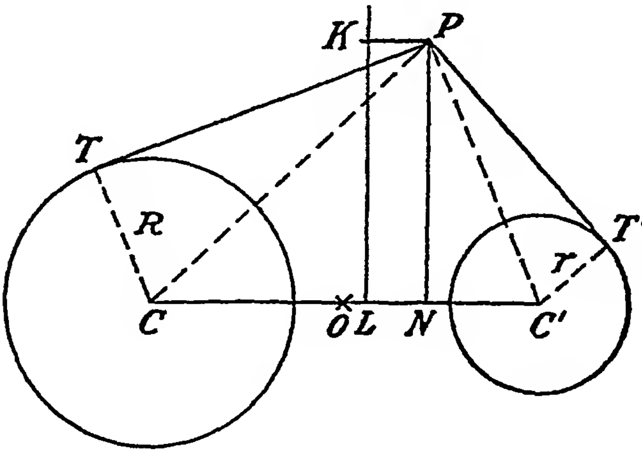
8 Circles are described on the sides of a triangle as diameters prove that the radical centre of these circles is the orthocentre of the triangle

9 Circles are described on the joins of the orthocentre and vertices of a triangle as diameters, prove that the orthocentre of the triangle is the radical centre of these circles

10 Prove that the radical axis of two circles bisects their four common tangents

11 *The difference of the squares of the tangents to two circles, from*

any point P, is equal to twice the rectangle contained by the join of their centres and the perpendicular from P on their radical axis



Draw PN, PK perpendicular to the line of centres and the radical axis LK respectively

Let O be the middle point of CC'

Then

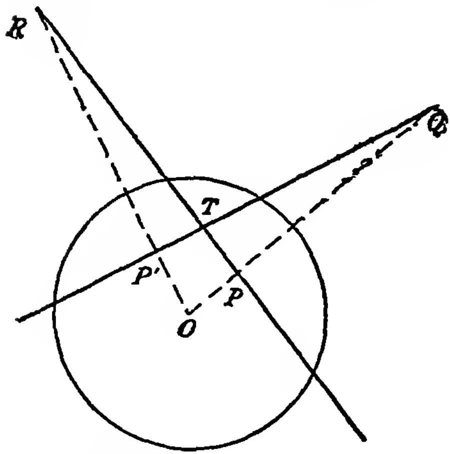
$$\begin{aligned} PT^2 - PT'^2 &= (CP^2 - R^2) - (C'P^2 - r^2) \\ &= (CP^2 - C'P^2) - (R^2 - r^2) \\ &= 2ON \cdot CC' - 2OL \cdot CC' \qquad [\text{Prop } II \\ &= 2LN \cdot CC' \\ &= 2PK \cdot CC' \end{aligned}$$

Poles and Polars

DEF.—*If on the radius of a circle whose centre is O and radius r two points P and Q be taken on the same side of O such that*

$$OP \cdot OQ = r^2,$$

then P and Q are said to be **inverse points** *with respect to the circle*



Since P', R are inverse points,

$P'Q$ is the polar of R

Note—Two points which are so related that the polar of each passes through the other are called **conjugate points**

7 Show that any pair of inverse points with respect to a circle are conjugate points

8 *If a straight line S passes through the pole of another straight line S′, then S′ passes through the pole of S*

For in the figures of this proposition TT' passes through the pole Q of PL, and PL passes through the pole L of TT'

Note—Two such lines as PL and TT' are called **conjugate lines**

9 *The polars of collinear points are concurrent, the point of concurrence being the pole of the line on which they lie*

For let the points A, B, C, etc, lie on the line L, whose pole is P

Then the polar of P passes through A, B, C, etc, therefore, by Ex 6, the polars of A, B, C, etc, pass through P

10 *If a line pass through a fixed point its pole lies on a fixed straight line*

For it lies on the polar of the fixed point

11 *The intersection of the polars of two points is the pole of their join*

For in the figure of Ex 6 the polar of Q passes through T, hence the polar of T passes through Q Similarly, it passes through R Therefore QR is the polar of T

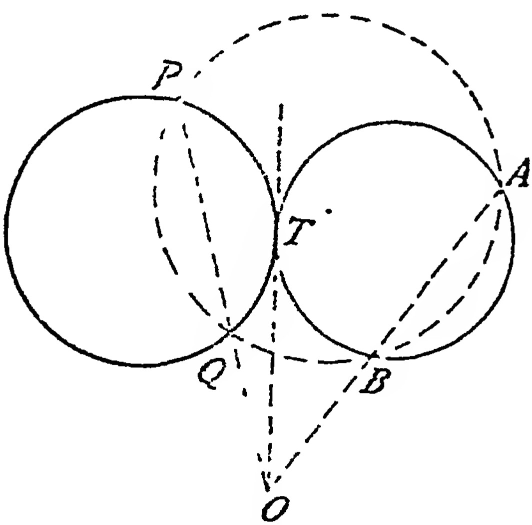
12 *The join of the poles of two lines is the polar of their point of intersection*

The Tangencies

We shall now consider a few simple constructions in which a circle is required to be drawn so as to pass through given points and touch given lines and circles These problems are known as *the tangencies*, and in each of them the required circle is made to satisfy three specified conditions

Proposition IV

Draw a circle to pass through two given points and to touch a given circle



Let P, Q be the given points and BAT the given circle

Draw any circle passing through P and Q and cutting the given circle in A, B Produce AB, PQ to meet in O, and from O draw OT a tangent to the given circle.

Draw a circle through P, Q, and T

Then PQT is the required circle

For, A, B, Q, P are concyclic,

$OB \cdot OA = OQ \cdot OP$ [Bk. I, Prop. 44.

Again, OBA is a secant and OT a tangent,

$\therefore OB . OA = OT^2$ [Bk. I, Prop. 44, Cor. 1.

Hence $OQ \cdot OP = OT^2$,

OT touches the circle PQT.

[Bk. I, Prop. 44, Cor. 2.

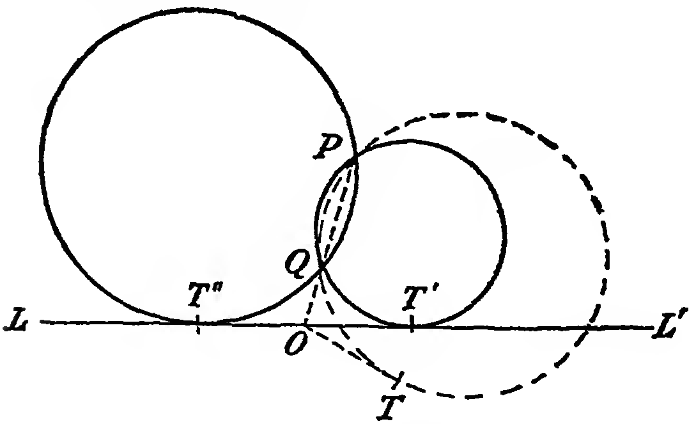
Thus the two circles, having a common tangent at T, touch one another at T.

Therefore the circle PQT is drawn as required.

Since two tangents can be drawn from O to the given circle, there are two solutions.

PROPOSITION V

Draw a circle to pass through two given points and to touch a given straight line.



Let P and Q be the given points and LL' the given line. Suppose the circle PQT' is drawn as required.

Produce PQ to meet the given line in O, then O is a fixed point.

And $OT'^2 = OQ . OP$ [*Bk I, Prop* 44, *Cor* 1

Hence OT' is constant.

To obtain OT' we may find the side of a square equal to the known rectangle OQ OP [*Bk II, Prop* 76

But it is better to proceed as follows —Draw any circle through P and Q; from O draw OT a tangent to this circle. From the given line cut off OT', OT'' on either side of O, equal to OT.

The circumcircles of the triangles PQT', PQT'' are the required circles

For ∵ OT is a tangent and OQP a secant,

∴ $OQ . OP = OT^2$. [*Bk I, Prop.* 44, *Cor.* 1

Hence OQ $OP = OT'^2$,

and OQ $OP = OT''^2$.

Therefore the line LL' touches the circles at T', T''.

[*Bk I, Prop* 44, *Cor* 2.

Note.—When PQ is parallel to LL' this construction fails In that case let the right-bisector of PQ meet LL' in T. Then PQT is the required circle.

Proposition VI

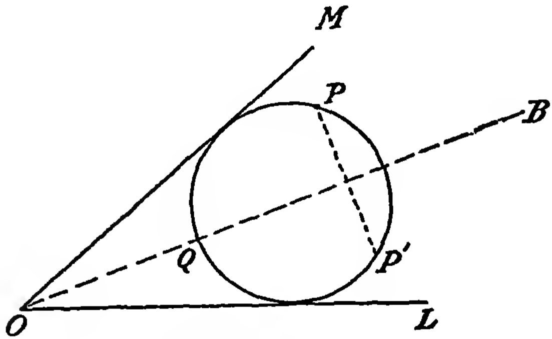
Draw a circle to pass through a given point and to touch two given straight lines

Let P be the given point and OL, OM the given straight lines

Suppose the circle $PP'Q$ is drawn as required

The centre of the circle must lie on the bisector OB of the angle between OL and OM

Also the chord of the circle through P, which is perpendicular to OB, is bisected by OB



Hence we have the following construction —

Draw the bisector of the angle between the given lines

Find the image P' of the given point in this bisector, then this image is a second point on the required circle. Draw a circle to pass through P, P', and to touch either OL or OM [*Prop* V.

As in Prop V, there will be two solutions

Proposition VII

Draw a circle to touch two given straight lines and a given circle

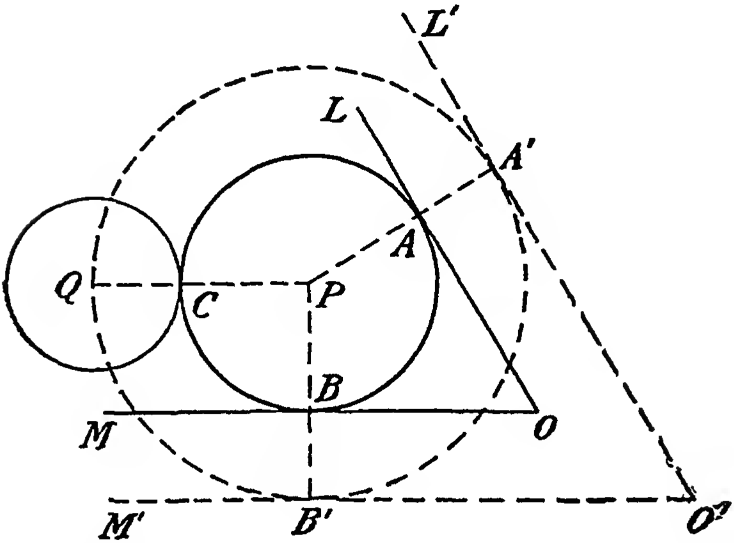
Let OL, OM be the given straight lines and Q the centre of the given circle.

Draw $O'L'$, $O'M'$ ∥ OL, OM at the distance of the radius of the given circle and on the sides of OL, OM remote from Q.

Draw a circle, centre P, touching $O'L'$, $O'M'$ and passing through Q [*Prop* VI

Let PQ cut the given circle in C.

With centre P and radius PC describe the circle ABC.



Then ABC is the required circle Let the student supply the necessary analysis and proof

Since there are two solutions to Prop VI, there will be two solutions here also

Again, the parallels can be drawn on the sides of OL, OM nearest to Q. This will give two more solutions Thus there are four solutions

Proposition VIII

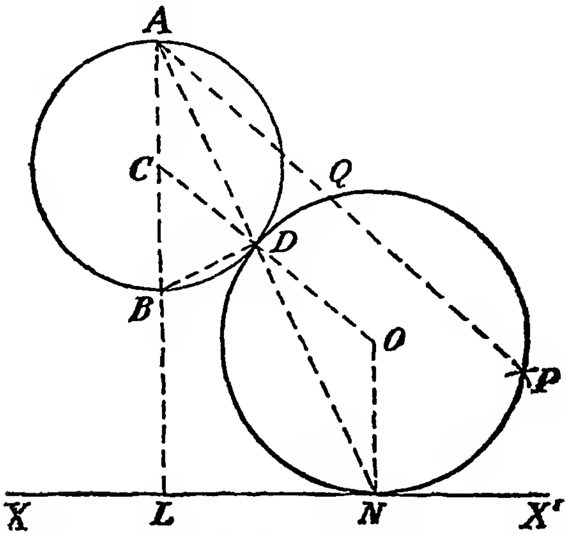
To draw a circle to pass through a given point and to touch a given straight line and a given circle

Let P be the given point, XX' the given straight line, and C the centre of the given circle

Suppose the circle QPN is drawn as required.

Through C draw $CL \perp XX'$, cutting the given circle in A, B

Through the centre O of the circle QPN draw $ON \perp XX'$.



Join OC, passing through D, the point of contact of the circles

Join BD, AD, ND, and let AP cut the circle NPQ in the point Q

$$ON \parallel AC,$$
$$\angle ACD = \angle DON$$

In the two isosceles $\triangle$s ACD, DON the vertical angles are equal,

the angles at their bases are also equal

Hence $\angle ODN = \angle CDA$,

$\therefore$ ADN is a st line.

Now, $\because$ $BLND$ is cyclic,

$\therefore$ $AB \cdot AL = AD \cdot AN$ [Bk I, Prop 44.

And $\because D, N, P, Q$ are concyclic,

$\therefore AD \cdot AN = AQ \cdot AP$. [*Bk. I, Prop.* 44.

Hence $AQ \cdot AP = AB \cdot AL$,

$$\text{or } \frac{AP}{AB} = \frac{AL}{AQ}.$$

Thus AQ is a fourth proportional to three known lines AP, AB, and AL

Hence we have the following construction —

Through C draw a perpendicular CL to the given line XX', cutting the given circle in A and B.

Find a fourth proportional to AP, AB, and AL

From AP cut off AQ equal to this fourth proportional

Draw a circle to pass through the points P, Q and to touch the line XX'. [*Prop.* V

Then this is the required circle

The student will supply the proof.

Since there are two solutions to Prop V., there will be two solutions here also

Again, another two solutions may be obtained by joining BP instead of AP. Thus there are four solutions

PROPOSITION IX

Draw a circle to touch two given circles and a given straight line.

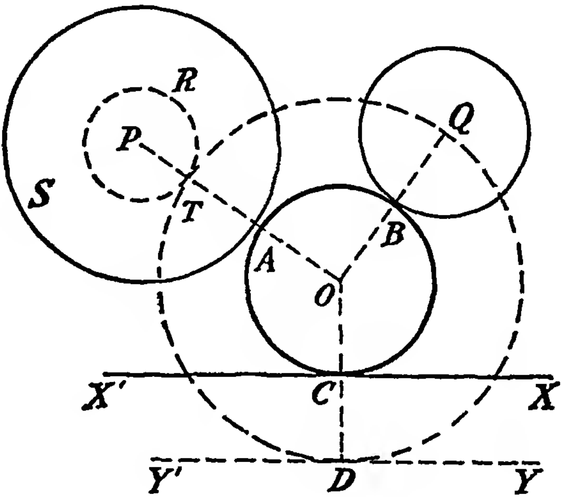
Let P, Q be the centres of the given circles and $X'X$ the given straight line

Suppose the circle ABC, centre O, is drawn as required; and let it touch the given circles at A and B and the given straight line at C

With centre O and radius OQ, Q being the centre of the lesser circle, describe a concentric circle, and with

centre P and radius equal to $PA - QB$ describe the circle TRS Then these two circles will touch at some point T

Produce OC to meet the circumference of the circle TQD in D Through D draw $Y'DY$ parallel to $X'CX$



Then $Y'Y$ touches the circle TQD in D

Now notice that CD is equal to BQ, the radius of the lesser given circle, and PT is equal to the difference of the radii of the given circles

Hence with the given data the line YY' and the circle TRS can be drawn And when the line YY' and the circle TRS have been drawn, the circle TQD can be constructed, for it passes through a given point Q, touches a given line YY' and a given circle TRS [*Prop* VIII

Finally, when the circle TQD has been drawn, the circle ABC can be described For join OQ, cutting the

circle, centre Q, in B Then OB is the radius of the required circle

The student can now write out the construction and proof

There are four solutions to Prop VIII, therefore there are four solutions in this case also Moreover, the line $Y'Y$ may be drawn on the same side of $X'X$ as Q, this will give another four solutions

Hence in this case there are eight solutions

Note —The tangencies include ten problems We have given six here, and two were given in Bk. II, Prop 68 and Props 69, 70 Two more will be found among the exercises in Section V.

SECTION V—SIMILAR FIGURES

Homothetic Figures

PROPOSITION I

IF any point O be joined to the vertices of a polygon ABCD, and if on the straight lines OA, OB, OC, OD, or their prolongations through O, points a, b, c, d be taken such that

$$\frac{Oa}{OA} = \frac{Ob}{OB} = \frac{Oc}{OC} = \frac{Od}{OD},$$

then the polygon abcd will be similar to the polygon ABCD

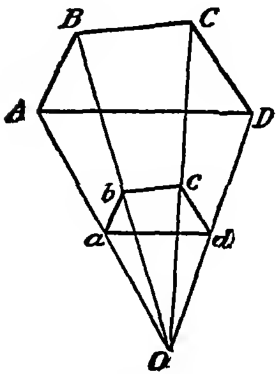


FIG. 1.

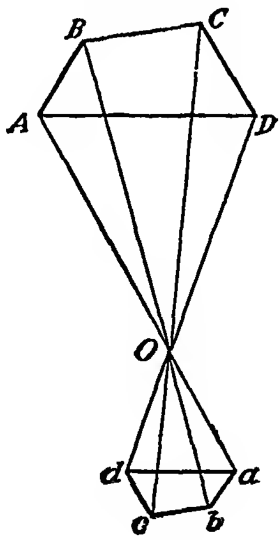


FIG 2.

$$\because \frac{Oa}{OA} = \frac{Ob}{OB} = \frac{Oc}{OC}, \qquad [Hyp$$

$\therefore$ $ab \parallel AB$, and $bc \parallel BC$. [*Bk I, Prop.* 45.

Hence $\angle abc = \angle ABC$.

In the same way we can prove that each angle of *abcd* is equal to the corresponding angle of *ABCD*.

Therefore the polygons are equiangular.

Again, $\because ab \parallel AB$,

$\therefore$ Δs *Oab*, *OAB* are equiangular.

Hence $$\frac{ab}{AB} = \frac{Oa}{OA} \qquad [Bk. I., Prop. 46.$$

Similarly, $$\frac{bc}{BC} = \frac{Ob}{OB}$$

Therefore $$\frac{ab}{AB} = \frac{bc}{BC},$$

i e $$\frac{ab}{bc} = \frac{AB}{BC}.$$

Hence the sides about the equal angles *b*, *B* are proportional

Similarly, the sides about each of the other equal angles are proportional

Therefore the polygons *abcd*, *ABCD* are similar

Note.—In Fig 1 the two polygons are said to be **similar and similarly situated**, and the point *O* is called their **external centre of similitude.**

Notice that in this case the pairs of corresponding sides of the two polygons (*ab*, *AB*), (*bc*, *BC*), etc., are drawn in the same sense

In Fig 2 the two polygons are said to be **similar and oppositely situated**, and the point *O* is called their **internal centre of similitude.**

In this case the pairs of corresponding sides (ab, AB), (bc, BC), etc, are drawn in opposite senses

DEF —*Two similar and similarly situated figures are said to be* **homothetic**, *and their external centre of similitude is called their* **homothetic centre.**

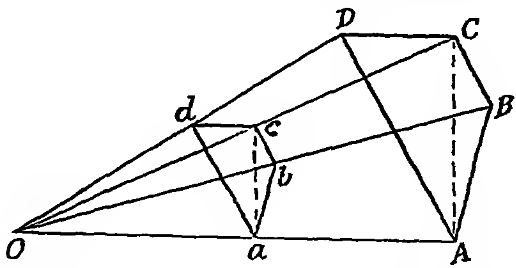
DEF —*Two similar but oppositely situated figures are said to be* **antihomothetic**, *and their internal centre of similitude is called their* **antihomothetic centre**

Thus the two polygons in Fig 1 are homothetic, and O is their homothetic centre, while those in Fig 2 are antihomothetic, and O is their antihomothetic centre.

Notice that in homothetic figures the joins of corresponding angular points, being produced, meet in the homothetic centre, whereas in antihomothetic figures the joins of corresponding angular points cross one another at the antihomothetic centre

PROPOSITION II

If two similar polygons be placed so that their corresponding sides are parallel and drawn in the same sense, then they shall have an external centre of similitude



Let the similar polygons $abcd$, $ABCD$ be placed so that the sides ab, bc, cd, da are parallel to the sides AB, BC, CD,

DA respectively, and are drawn in the same sense; then shall Aa, Bb, Cc, Dd produced meet in the same point O.

Let Aa, Bb produced meet in O.

Join OC, Oc, ac, AC.

$\because$ $\triangle$s abc, ABC are similar, [*Bk I*, *Prop.* 48.

$$\therefore \angle Oac = \angle OAC,$$

and
$$\frac{ac}{AC} = \frac{ab}{AB} = \frac{Oa}{OA},$$

i e.
$$\frac{ac}{Oa} = \frac{AC}{OA}.$$

Hence $\triangle$s Oac, OAC are similar. [*Bk. I.*, *Prop.* 48.

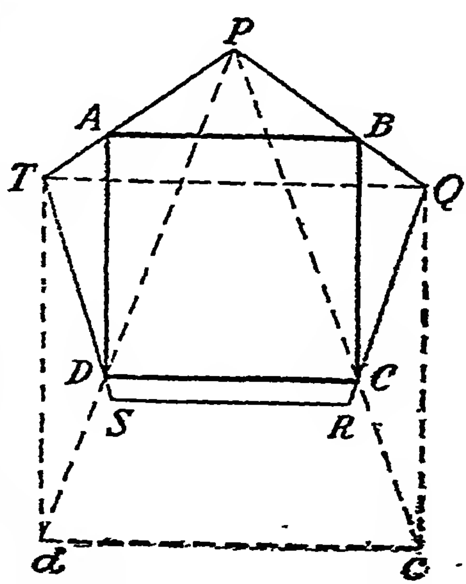
Therefore $\angle aOc = \angle AOC$.

Hence C, c, O are in one straight line.

Similarly. D, d. O are in one straight line.

Thus the polygons $abcd$, $ABCD$ are homothetic.

COR.—*If two similar polygons be placed so that their corresponding sides are parallel but drawn in opposite senses, then they shall have an internal centre of similitude*, i.e. *they shall be antihomothetic.*



EXERCISES

1. *In a given regular pentagon inscribe a square, so that one side of the square may be parallel to a side of the pentagon.*

Let $PQRST$ be the given pentagon; and suppose the square $ABCD$ constructed as required.

On TQ describe the square $TQed$

Then $ABCD$ and $TQed$ are homothetic, and P is their homothetic centre, for the joins of the corresponding angular points (B, Q) and (A, T) meet in P.

Hence the joins of the two other pairs of corresponding angles must also meet in P

Therefore dD, eC pass through P.

From the above analysis we have the following construction:—

On TQ describe the square $TQed$, join Pd, Pe, cutting the pentagon in the points D, C

Then DC is a side of the required square.

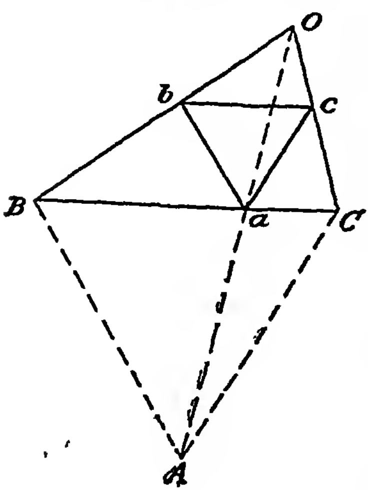
Supply proof

2 In a given hexagon inscribe a square, so that one side of the square may be parallel to a side of the hexagon

3 In a given equilateral triangle inscribe a regular octagon, so that one side of the octagon may be along a side of the triangle and two angular points on the other two sides of the triangle

4 In a given equilateral triangle inscribe a regular hexagon

5 In a given square inscribe a regular pentagon, so that one of its angular points may lie on a given diagonal of the square, and the other four on the four sides of the square



6 *Inscribe an equilateral triangle in a given triangle, having one side parallel to a given side of the triangle*

Let abc be the equilateral triangle constructed as required, having its side bc parallel to the side BC of the given triangle OBC

On BC describe the equilateral triangle ABC

Then Δs ABC, abc are clearly homothetic, and O, the vertex of the given triangle, is their homothetic centre

Supply construction and proof

7 With sides 3″, 2 8′, and 2″ draw a triangle, in it inscribe an equilateral triangle having one of its sides parallel to the longest side of the triangle

8 In a given triangle inscribe a square

9 In a given triangle inscribe a triangle similar to a given one, and having one of its sides parallel to a given straight line

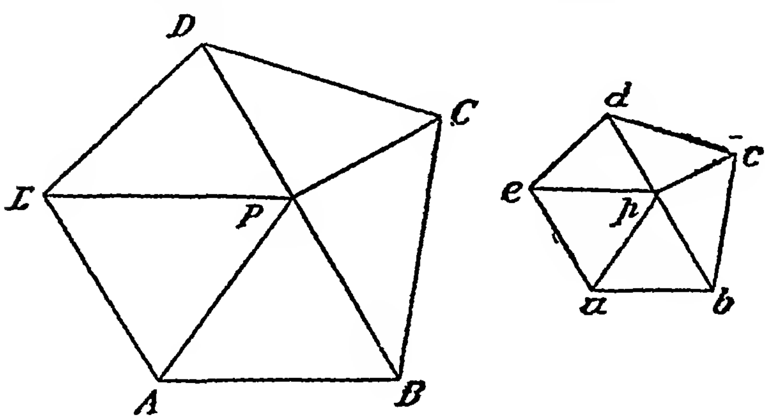
10 In a given triangle inscribe a triangle similar to a given one, and having one of its sides parallel to a side of the given triangle

11 With sides 3 cm, 4 cm, and 2 6 cm draw a triangle: in it inscribe an equilateral triangle having one of its sides inclined at an angle of 45° to the longest side

12 With sides 21 mm, 25 mm, and 39 mm draw a triangle, and in it inscribe a rhombus having one of its angles of 120° and its base along the longest side of the triangle

Proposition III

If a polygon is divided into triangles by lines joining a point to its vertices, any similar polygon can be divided into the same number of corresponding similar triangles.



Let $ABCDE$ be a polygon divided into five triangles by straight lines drawn from P to the vertices, then the similar polygon *abcde* can be divided into five corresponding similar triangles

Make $\angle$s bap, $abp = \angle$s BAP, ABP.

Then $\triangle$s PAB, pab are equiangular, and consequently similar,

$$\therefore \frac{bp}{ab} = \frac{BP}{AB} \qquad (1)$$

But $\because$ the polygons are similar,

$$\frac{ab}{bc} = \frac{AB}{BC} \qquad (2)$$

From (1) and (2) we get

$$\frac{bp}{bc} = \frac{BP}{BC},$$

and $$\angle pbc = \angle PBC,$$

$\therefore$ $\triangle$s pbc, PBC are similar [*Bk I*, *Prop* 48

Similarly, other pairs of triangles can be proved similar.

Proposition IV

The ratio of the areas of similar polygons is equal to the ratio of the squares of corresponding sides

We have $$\frac{\triangle PAB}{\triangle pab} = \frac{AB^2}{(ab)^2},$$

$$\frac{\triangle PBC}{\triangle pbc} = \frac{BC^2}{(bc)^2} = \frac{AB^2}{(ab)^2},$$

$$= \qquad = \frac{AB^2}{(ab)^2},$$

$$\therefore \frac{\text{Sum of numerators}}{\text{Sum of denominators}} = \frac{AB^2}{(ab)^2},$$

$$\frac{\text{Polygon } ABCDE}{\text{Polygon } abcde} = \frac{AB^2}{(ab)^2}$$

EXERCISES

1 In the figure of Prop III prove that the lines from P to the vertices of $ABCDE$ are proportional to the lines from p to the vertices of $abcde$

DEF —**Corresponding points** *in the planes of two similar figures are such that lines drawn from them to the vertices of the two figures are in the ratio of the corresponding sides of the two figures*

Thus P, p are corresponding points

2 Prove that the circumcentres, the incentres, the orthocentres, the centroids, and the midcentres of two similar triangles are pairs of corresponding points

3 In Prop III show how the two figures can be placed so that the point P is their homothetic centre

4 Prove that any two circles are similar figures.

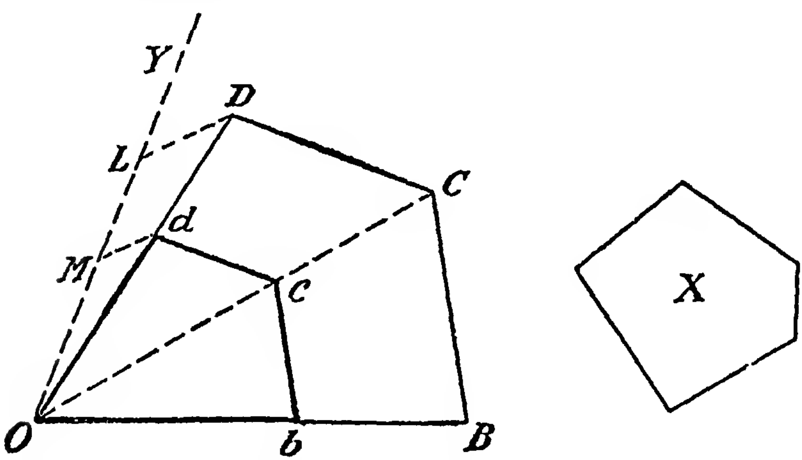
5 The perimeters of similar figures are proportional to their corresponding sides

6 The circumferences of circles are in the ratio of their radii

7 The areas of circles are to one another as the squares of their radii

PROPOSITION V

To construct a polygon similar to one given polygon OBCD, and equal in area to another X.



Construct squares equal in area to the polygons $OBCD$ and X. [*Bk II, Prop* 77

Through O draw any line OY, and cut off OL, OM

equal to the sides of the squares which are equivalent to $OBCD$ and X respectively

Join LD, OC.

Draw Md, dc, $cb \parallel LD$, DC, CB

Then $Obcd$ is the required polygon For $OBCD$, $Obcd$ are homothetic [Prop I

$$\frac{\text{Area } Obcd}{\text{Area } OBCD} = \frac{Od^2}{OD^2} \qquad \text{[Prop IV}$$

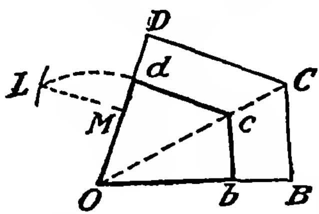
$$= \frac{OM^2}{OL^2} \qquad \text{[Bk I, Prop 45}$$

$$= \frac{\text{Area } X}{\text{Area } OBCD} \qquad \text{[Const}$$

Hence the area $Obcd$ = area X, and the figure $Obcd$ is similar to the figure $OBCD$

EXERCISES

1 *Describe a polygon similar to a given polygon, but of half its area*



Let $OBCD$ be the given polygon Bisect OD in M, and draw $ML \perp OD$ and equal to OM Cut off $Od = OL$

Join OC

Draw dc, $cb \parallel DC$, CB

Then $Obcd$ is the required polygon For if $OD = 2$, then $OM = 1$ and $OL = \sqrt{2}$

Hence $$\frac{Od}{OD} = \frac{\sqrt{2}}{2}$$

Complete the proof

2 *Bisect a triangle by a line parallel to one of its sides*

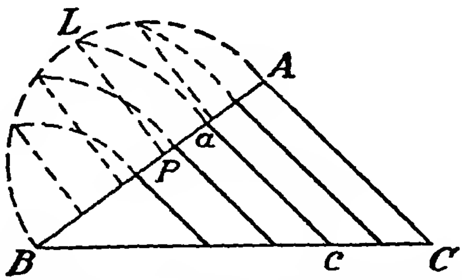
Let OCD be the Δ

The line dc, drawn as above, bisects it

3 With sides 32 mm , 35 mm , and 41 mm construct a triangle, and bisect it by a line drawn parallel to the shortest side

4 Draw a triangle with sides 2″, 2 5″, and 3 2″, bisect it by a line drawn parallel to the longest side

5 *Divide a triangle into any number of equal parts (say five) by lines parallel to one of its sides*



On the side AB describe a semicircle Divide AB into five equal parts Let P be one of the points of division Through P draw $PL \perp AB$ meeting the semicircle in L With centre B and radius BL describe an arc cutting AB in a, through a draw $ac \parallel AC$.

Treat the remaining points of division in the same way

Now, $$\frac{\Delta Bac}{\Delta BAC} = \frac{(Ba)^2}{BA^2} \qquad [Bk\ I, Prop\ 50$$
$$= \frac{BL^2}{BA^2} \qquad [Const$$
$$= \frac{BP \cdot BA}{BA^2} \qquad [Bk\ I, Prop\ 27, Cor\ 1$$
$$= \frac{BP}{BA}$$
$$= \tfrac{2}{5}, \qquad [Const$$
$$\therefore \Delta Bac = \tfrac{2}{5}\Delta BAC$$

In the same way it can be shown that the Δs whose vertices are at B are equal in area to $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{5}$, and $\frac{5}{5}$ of ΔBAC respectively

6 Draw a triangle with sides 2 3″, 2 5″, and 3″, divide it into three equal parts by lines drawn parallel to the side 2 5″

7 With sides 7 cm, 8 cm, and 9 cm draw a triangle, and divide it into seven equal parts by lines parallel to the shortest side

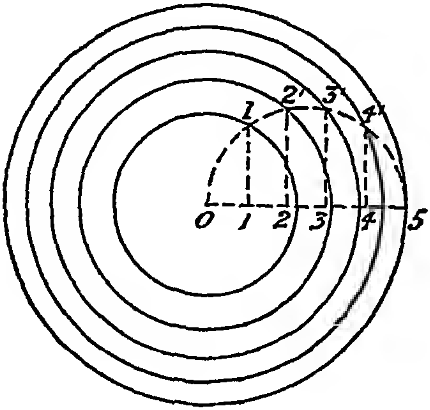
8 *Draw a quadrilateral similar to a given quadrilateral $BACD$, but of three-fifths its area*

In BA find the point P as in Ex 5 With BP as one side draw a quadrilateral homothetic with $BACD$ (See Ex 1)

9 *Divide a circle into any number (say five) of equal concentric rings*

On the radius $O5$ describe a semicircle. Divide $O5$ into five equal

parts, and through the points of division 1, 2, 3, 4 draw perpendiculars to meet the semicircle in 1′, 2′, 3′, 4′



With centre O and radii $O1'$, $O2'$, $O3'$, $O4'$ describe circles

Proceed with the proof as in Ex 5

10 From a circle of 3 cm radius cut off a ring containing one third the area of the circle

11 Describe a circle whose area is four-sevenths of the area of a circle of 1 in radius.

12 Describe an equilateral triangle equal to a given square

13 Describe an equilateral triangle containing an area of 4 sq in

14 Describe a square equal to an equilateral triangle of 1 in side

15 *Describe an equilateral triangle equal to a given triangle*

Let OPQ' be the given Δ

On OP describe an equilateral triangle OPQ, and on OQ describe the semicircle OQL.

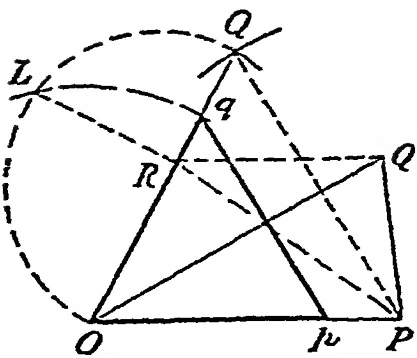
Draw $Q'R \parallel OP$, and $RL \perp OQ$

Cut off $Oq = OL$, and draw $qp \parallel QP$

Then Opq is the required equilateral triangle

For $$\frac{\Delta Opq}{\Delta OPQ} = \frac{Oq^2}{OQ^2} \qquad \text{[}Bk\ I, Prop\ 50.$$

$$= \frac{OL^2}{OQ^2} \qquad \text{[}Const.$$



$$= \frac{OR . OQ}{OQ'} \qquad \text{[Bk. I, Prop. 27, Cor. 1}$$

$$= \frac{OR}{OQ}$$

$$= \frac{\Delta OPR}{\Delta OPQ} \qquad \text{[Section I., Prop. VII.}$$

$$= \frac{\Delta OPQ}{\Delta OPQ}, \qquad \text{[Bk. I, Prop. 21.}$$

$\therefore \Delta Opq = \Delta OPQ'$,

and ΔOpq is evidently equilateral.

16. With sides 3", 3.2", and 2" draw a triangle, and construct an equilateral triangle of equivalent area.

17. Construct a regular hexagon of 2.25 sq. in. area.

18. Construct a regular octagon containing an area of 4 sq. in.

19. Construct a regular pentagon containing an area of 1.44 sq. in.

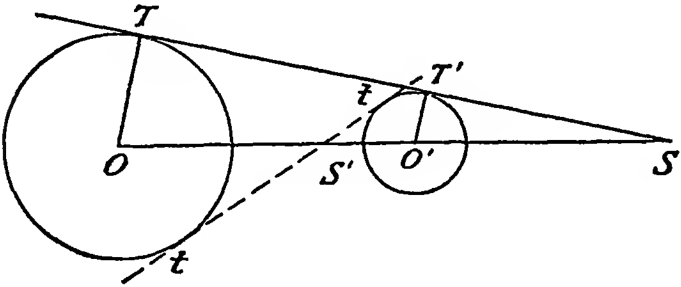
20. Draw a regular hexagon of 1 in. side, and construct another containing half its area.

Centres of Similitude of Two Circles

DEF.—*If the line joining the centres of two circles be divided internally and externally in the ratio of their radii, the points of division are called the* **internal** *and the* **external centres of similitude** *of the circles.*

PROPOSITION VI

The direct and transverse common tangents of two circles cut the line of centres in their external and internal centres of similitude respectively



Let the direct common tangent TT' meet the line of centres in S

Then from similar Δs OTS, $O'T'S$ we have

$$\frac{OS}{O'S} = \frac{OT}{O'T'}$$

Thus the line OO' is divided externally in S in the ratio of the radii of the circles

Similarly, we can prove that the transverse common tangent tt' divides OO' internally at S' in the ratio of the radii

Hence S, S' are the external and internal centres of similitude

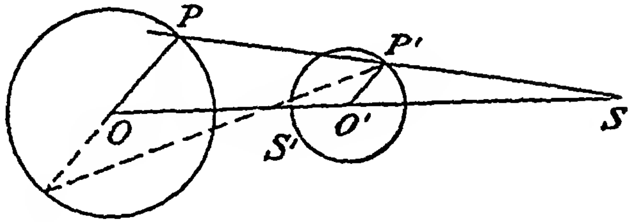
EXERCISES

1 *The line joining the extremities of parallel radii of two circles passes through their external, or internal, centre of similitude, according as the radii are drawn in the same or opposite senses*

Let OP, $O'P'$ be parallel radii, and let PP', OO' meet in S, then from similar Δs

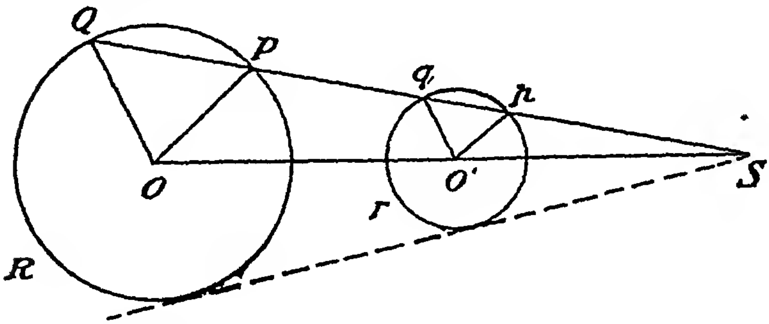
$$\frac{OS}{O'S}=\frac{OP}{O'P'}$$

Hence S is the external centre of similitude



Similarly, prove that the join of the extremities of the parallel radii Op, $O'P'$, drawn in opposite senses, passes through the internal centre of similitude

2 *If through a centre of similitude of two circles a secant be drawn to both the circles, the radii drawn to a pair of corresponding points are parallel, and the distances of these points from the centre of similitude are in the ratio of the radii of the circles*



In Δs SOP, $SO'p$

$$\frac{SO}{SO'}=\frac{OP}{O'p}, \qquad [Def.$$

and $\angle OSP=\angle O'Sp$, the remaining angles being both acute ;

the Δs are similar

Hence $OP \parallel O'p$, and $\frac{Sp}{SP}=\frac{O'p}{OP}$.

Similarly, $OQ \parallel O'q$, and $\dfrac{Sq}{SQ}=\dfrac{O'q}{OQ}$

3 *If S is a given point and PQR a given circle, prove that the locus of a point p which divides SP in a given ratio is a circle*

In the figure of the last exercise

$$\frac{Sp}{SP}=\textit{ratio of radii},$$

and the locus of p is the circle pqr

Note.—This is a very useful theorem

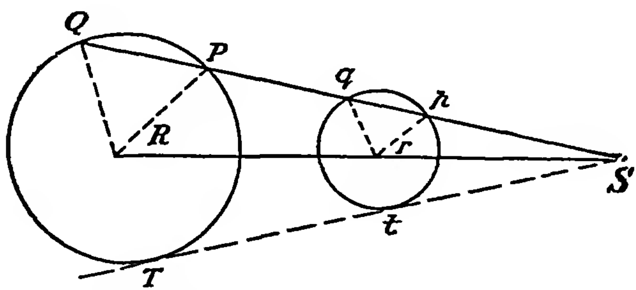
4. The base of a triangle and the median which bisects the base being given, prove that the locus of the centroid is a circle

5 The base of a triangle and the vertical angle being given, prove that the loci of the centre of gravity and the midcentre are circles

Proposition VII

If through S, a centre of similitude of two circles, a secant SpqPQ be drawn, cutting one of them in the points p, q, and the other in the corresponding points P, Q, then the rectangles

Sp SQ and Sq SP are constant



Let R, r be the radii of the circles Then

$$\frac{r}{R}=\frac{Sp}{SP}; \qquad \text{[}\textit{Prop VI, Ex. 2.}$$

$$\therefore \frac{r}{R} = \frac{Sp \cdot Sq}{SP \cdot Sq}$$
$$= \frac{St^2}{SP \cdot Sq}.$$

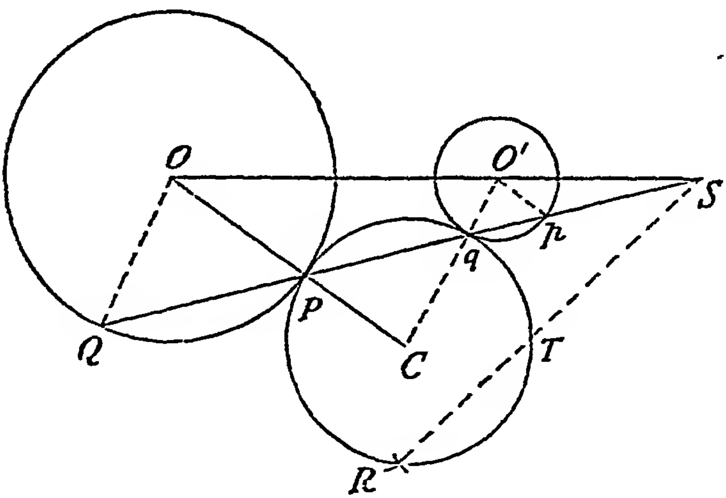
[*Where St is a tangent through S*

Hence $Sq \cdot SP$ is a fourth proportional to three constant quantities R, r, St^2, and is therefore itself a constant.

Similarly, $Sp \cdot SQ$ is a constant.

EXERCISES

1 *If a circle touch two given circles the line through the points of contact passes through one of the centres of similitude of the given circles.*



Let O, O' be the given circles, and C the circle which touches them at the points P, q, and let Pq meet OO' in S Then

$$\angle OPQ = \angle CPq = \angle CqP = \angle O'qp = \angle O'pq,$$
$$\therefore OP \parallel O'p$$

Hence Pqp passes through a centre of similitude

[*Prop VI, Ex. 2.*

R

2 *Draw a circle to pass through a given point and to touch two given circles*

Let R be the given point and O, O' the centres of the given circles (figure of last Ex)

Suppose $PqTR$ is the required circle, and let it cut SR in T

Then $ST \cdot SR = Sq \cdot SP =$ Constant, [*Ex* 1

$\therefore$ T is a known point

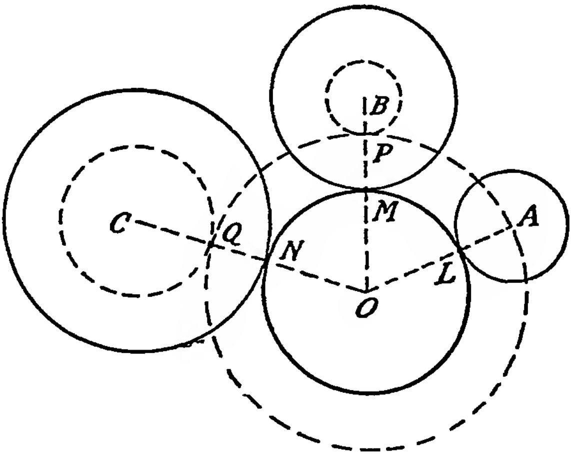
If we draw a circle through R and T touching one of the given circles [*Section IV*, *Prop IV*], it will touch the other given circle

Hence the following construction —

Find S the external centre of similitude of the given circles. Draw St a tangent to the lesser circle, and join SR From SR cut off ST such that $ST \cdot SR = St^2$ Draw a circle through R and T to touch one of the given circles

Since there are two solutions of Section IV, Prop IV, hence there are two solutions in this case Again, instead of S we may take S' the internal centre of similitude, this will give rise to two more solutions. Thus there are four solutions.

3 *To draw a circle to touch three given circles*



Let A, B, C be the centres of the given circles, and r_1, r_2, r_3 their radii respectively

Suppose r_1 is not greater than either r_2 or r_3.

With centre B and radius $r_2 - r_1$ describe the circle BP, also, with centre C and radius $r_3 - r_1$ describe the circle CQ

Draw a circle APQ to pass through A and to touch the circles BP, CQ in P and Q [Ex 2] Let O be the centre of this circle

Join OA, cutting the circle, centre A, in L

With O as centre and OL as radius describe the circle LMN.

Then LMN is the required circle

The student can easily supply the proof

MISCELLANEOUS EXAMPLES.—VII.

1 Two circles whose centres are A and B touch externally at C; prove that if PQ be a common tangent not passing through C, then

$$PC^2 + QC^2 = 4AC \cdot BC$$

2 A circle touches one side BC of a triangle, and the other sides AB, AC produced, the points of contact being D, F, E If I be the centre of the inscribed circle, prove that the triangles IAE, IAF are together equal to the triangle ABC

3 If a quadrilateral be inscribed in a circle, and from any point on the circumference perpendiculars be drawn to the four sides, their lengths are the four terms of a proportion

4 AB, CD are diameters of a circle at right angles to each other. BPQ is a straight line meeting the circle in P and CD in Q

Show that $$CP \cdot DP = BP \cdot QP$$

5 ABC are three points in a line Circles are described on AB, BC, AC as diameters If the common tangent at B meets the circle on AC at P, prove that QPR is a right angle where QR is a common tangent to the circles on AB and BC

6 If two intersecting circles, one of which is of given radius, touch a fixed straight line at the fixed points A and B, and if P and Q be the points of contact of the other common tangent, find the locus of the middle point of PQ.

7 Prove that the difference of the squares of the direct and transverse common tangents to two circles is equal to the product of their diameters

8 ABC and EBC are two isosceles triangles, E lying on AB A is the vertex of one and C of the other Show that the rectangle $AB \cdot BE =$ the square on BC

9 Prove that the triangle formed by joining the centres of the escribed circles of a triangle is similar to that formed by joining the points of contact of the inscribed circle

10 If the middle points of adjacent sides of a rectangle be joined, and four circles be inscribed in the triangles so formed, prove that the rectangle whose vertices are the centres of these circles is half the original rectangle.

11 If ABC be a triangle in which ACB is a right angle, show that the area of the square on AB, together with four times the area of the triangle, is equal to the area of the square on a line whose length is equal to AC and CB together

12 The locus of a point which moves in the plane of a triangle so that the sum of the squares of its distances from the angular points is constant is a circle

13 A variable circle touches two fixed circles Show that the line joining the points of contact passes through a fixed point

14 PQ is the common chord of any two equal circles which touch a given straight line in two given points A, B, show that PA, QB intersect on a fixed circle

15 If D be any point in the side BC of a triangle ABC, prove that

$$AB^2 \cdot CD + AC^2 \cdot BD = BC \cdot BD \cdot CD + AD^2 \cdot BC.$$

16 An equilateral triangle DEF is inscribed in a given equilateral triangle ABC so that D falls on BC, E on CA, and F on AB Prove that the area of DEF is a minimum when its sides are parallel to the sides of ABC

17 The circles $ABCD$ and $AEFD$ intersect in A and D, and the straight lines BAE and CDF are drawn, prove that BC and EF are parallel

18 Given the nine points circle of a triangle, one extremity of the base, and the straight line in which the base lies, construct the triangle

19 If one of two circles lies wholly within the other, find the points P on the inner circle and Q on the outer circle such that PQ may be (i) a minimum and (ii) a maximum, no part of PQ lying within the inner circle

20 If a triangle given in species have one vertex fixed and if a second vertex moves along a fixed line, then the third will also move along a fixed line [*When the angles of a triangle are given, it is said to be given in species*]

21 If an equilateral triangle have one vertex fixed, and if a second vertex moves along a fixed line, then the third will also move along a fixed line

22 A and B are two given points on the same side of a given line L It is required to find the point on L at which AB subtends the maximum angle.

23 In the last exercise show that there is a minimum position between two maxima.

24 Of all triangles on the same base and between the same parallels the isosceles triangle has the maximum vertical angle

25 A and B are two given points without a given circle C Find the point on C at which AB subtends the maximum angle

26 P is any point on the base BC of an isosceles triangle ABC; show that the radii of the circles APB, APC are equal

27 Prove that if two circles intersect, the tangents at a point of intersection are equidistant from a centre of similitude

28 If D, E, F be points on the sides BC, CA, AB of a triangle ABC, such that

$$BD^2+CE^2+AF^2=CD^2+AE^2+BF^2,$$

show that the straight lines drawn at right angles to the sides through D, E, F will meet in a point.

29 Equilateral triangles are described on the sides of a triangle, externally, prove that their circumcircles meet in a point and the centres are the vertices of an equilateral triangle.

30 Describe a triangle of given species so that one angular point may be at a given point and the others on given straight lines

31 If I_1, I_2, I_3 are the centres of the circles escribed to the sides BC, CA, AB of a triangle, and if I_1X, I_2Y, I_3Z are perpendicular to BC, CA, AB respectively, then I_1X, I_2Y, I_3Z are concurrent

32 Through a given point within a circle draw a chord of given length

33 Through a given point without a circle draw a straight line to cut the circle so that the part of it which is intercepted by the circle may have a given length

34 The base BC and the vertical angle A of a triangle being given, find the loci of I, I_1, I_2, I_3, the incentre, and the three excentres

35 In the triangle ABC, in which the angle at C is a right angle, the length of the perpendicular drawn from C on AB is p; prove that

$$\frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}$$

36 The rectangle contained by the perpendiculars from any point in the circumference of a circle on two tangents is equal to the square of the perpendicular, from the same point, on their chord of contact

37. PQR is a straight line parallel and equal to the base BC of a triangle ABC, meeting the sides in P and Q Show that the triangles BPQ, AQR are equal

38 Construct a triangle, having given its pedal triangle.

39 Two variable circles touch a given straight line at two fixed points A and B, and also touch one another at the point P Find the locus of P

40 Construct a triangle having given the inscribed circle and an escribed circle

41 Construct a triangle, having given two of its escribed circles.

42 Construct a triangle, having given its incentre and two of its excentres

43 Inscribe a square in a given semicircle, and prove that the square inscribed in a circle is to the square inscribed in the semicircle as 5 to 2

44 The areas of any two polygons, described about the same circle, are to one another as their perimeters

45 If a hexagon be inscribed in a circle, the continued product of the perpendiculars drawn from any point in the circumference to the odd sides is equal to the continued product of the perpendiculars to the even sides

46 If a triangle be inscribed in a circle, and tangents to the circle be drawn at its angular points, the continued product of the perpendiculars let fall from any point in the circumference on the sides of the triangle is equal to the continued product of the perpendiculars from the same point on the tangents

47 On the sides of a triangle ABC squares BCA_1A_2, CAB_1B_2, ABC_1C_2 are described external to the triangle, show that

$$AA_1^2+BB_1^2+CC_1^2=AA_2^2+BB_2^2+CC_2^2$$

48 In an equilateral triangle ABC, AD is drawn to cut BC in D, show that the square on AD is greater than the rectangle contained by BA, BD

49 Construct an equilateral triangle, having given its centre and one point on each of two of its sides

50 On each of the sides of an acute angled triangle a square is described, so that the triangle is outside all the squares Straight lines are drawn through the angular points perpendicular to the opposite sides of the triangle, and produced so as to divide each of the squares into two parts Show that each square is equal to the sum of the adjacent parts of the other two squares.

DEFINITIONS

Terms defined in the text are not given here

Acute angle.—An angle less than a right angle is called an *acute angle*

Acute-angled triangle.—A triangle which has three acute angles is called *acute-angled*

Adjacent angles.—When three straight lines are drawn from a point, if one of them be considered as lying between the other two the angles which it makes with the other two are called *adjacent angles*

Altitude.—*See* Base

Angle.—When two straight lines are drawn from a point they are said to form an *angle*

Arc.—Any part of the circumference of a circle is called an *arc.*

Arm of angle.—The two straight lines which contain an angle are called its *arms*

Axis of symmetry.—A straight line in the plane of a figure, which is such that if the figure is folded about it the two parts exactly coincide, is called an *axis of symmetry of the figure.*

Base.—Any side of a triangle may be selected as its *base,* and then the perpendicular upon it from the opposite vertex is called its **altitude.**

Similarly, any side of a parallelogram may be selected as its base, and then the perpendicular distance of its opposite side is called its altitude or height.

Centre—The point within a circle from which all straight lines drawn to its circumference are equal is called the *centre of the circle*

Chord—A straight line joining two points on the circumference of a circle is called a *chord* of the circle

Chord of an arc.—A straight line joining the extremities of an arc is called the *chord of the arc*

Chord of contact of tangents—If two tangents be drawn to a circle the straight line joining the points of contact is called the *chord of contact* of the two tangents

Circle—A plane closed line, which is such that all straight lines drawn to it from a fixed point within it are equal, is called a *circle*

Circumference—The same definition as that of a circle.

Circumscribed circle, Circumcircle—The circle which passes through all the angular points of a rectilineal figure is called its *circumscribed circle* or *circumcircle*

Circumradius—The radius of the circumcircle of a figure is called the *circumradius* of the figure

Circumcentre—The centre of the circumcircle of a figure is called the *circumcentre* of the figure

Complement, Complementary.—Two angles which are together equal to one right angle are called *complementary angles*, and each angle is called the *complement* of the other.

Concentric—Circles which have the same centre are called *concentric* circles

Congruent figures—Figures which can be made to coincide are called *congruent figures*

Corollary.—A further inference from facts which have already been proved is called a *corollary*

Corresponding sides.—The pairs of sides of equi-angular triangles, which are opposite to equal angles, are called *corresponding sides*

Corresponding vertices.—Pairs of vertices of two similar figures, at which the angles are equal, are called *corresponding vertices*, and pairs of sides lying between corresponding pairs of vertices are called **corresponding sides.**

Cyclic figure.—A rectilineal figure which is such that a circle can be drawn to pass through all its vertices is called a *cyclic figure*

Decagon.—A figure of ten sides is called a *decagon.*

Diagonal.—A straight line joining any two non-adjacent vertices of a rectilineal figure is called a *diagonal.*

Diameter.—A straight line drawn through the centre of a circle and terminated both ways by the circumference is called a *diameter*

Dodecagon.—A figure of twelve sides is called a *dodecagon*

Equal angles—*Equal angles* are such that if their vertices are made to coincide, their arms can also be made to coincide

Equiangular figures.—Two rectilineal figures are said to be *equiangular* when the angles of the one, taken in order, are equal to the angles of the other, taken in order.

Equiangular figure.—A figure is said to be *equi-angular* when all its angles are equal.

Equilateral figure.—A figure is said to be *equilateral* when all its sides are equal

Equilateral triangle.—An equilateral triangle is a triangle having three equal sides.

Excentre.—The centre of the escribed circle is called the *excentre*.

External bisector of angle.—The bisector of the supplement of an angle is called the external bisector of the angle.

Heptagon.—A figure of seven sides is called a *heptagon*.

Hexagon.—A figure of six sides is called a *hexagon*.

Hypotenuse.—In a right-angled triangle the side opposite to the right angle is called the *hypotenuse*.

Incentre.—The centre of the inscribed circle is called the *incentre*.

Inscribed circle.—If all the sides of a rectilineal figure touch a circle, the circle is said to be *inscribed* in the figure, and the figure is said to be *described* about the circle.

If a circle touch one side of a triangle and the other two sides produced, it is said to be **escribed** to the triangle.

Isosceles triangle.—A triangle which has two sides equal is called an *isosceles* triangle.

Kite.—A quadrilateral which has two pairs of equal adjacent sides is called a *kite*.

Line.—A line is that which has length, but neither breadth nor thickness.

Mean proportional.—If A, B, C are three lines, such that $A : B = B : C$, B is said to be the mean proportional between A and C.

Obtuse angle.—An angle greater than a right angle is called an obtuse angle.

Obtuse-angled triangle.—A triangle which has an obtuse angle is called an *obtuse-angled triangle*.

Octagon.—A figure of eight sides is called an *octagon*.

Parallel straight lines.—Two straight lines which are in the same plane, and do not meet however far they may be produced both ways, are said to be *parallel*

Parallelogram.—A quadrilateral whose opposite sides are parallel is called a *parallelogram*

Pentagon.—A figure of five sides is called a *pentagon.*

Perimeter.—The whole length of the boundary of a figure is called its *perimeter*

Figures which have equal perimeters are said to be **isoperimetrical.**

Perpendicular.—If two lines are at right angles to each other, either of them is said to be *perpendicular* to the other

Plane.—A surface, which is such that the straight line joining any two points in it lies wholly in the surface, is called a *plane*

Plane figure.—A figure which lies wholly in one plane is called a *plane figure*

Point —A *point* is that which has position but not magnitude

Polygon —A plane figure bounded by more than four sides is called a *polygon*

Quadrilateral —A *quadrilateral* is a plane figure bounded by four straight lines

Radius —A straight line drawn from the centre of a circle to meet the circumference is called a *radius*

Rectangle.—A *rectangle* is a parallelogram having one of its angles a right angle.

Rectilineal figure.—A figure bounded by straight lines is called a *rectilineal figure*

Reflex angle.—An angle greater than two right angles is called a *reflex angle*

Regular figure.—A figure which is both equilateral and equiangular is called a *regular figure*

Rhombus.—A quadrilateral, all of whose sides are equal, but whose angles are not right angles, is called a *rhombus*

Right angle.—If a straight line standing on another straight line makes the adjacent angles equal, each of these angles is called a *right angle*, and the straight lines are said to be **at right angles to each other.**

Right-angled triangle.—A triangle which has a right angle is called *right-angled*

Right-bisector.—The straight line which bisects a given straight line at right angles is called the *right-bisector* of the given straight line

Secant.—A straight line drawn to cut a circle is called a *secant*

Sector.—The figure bounded by an arc of a circle and the radii drawn to its extremities is called a *sector of a circle*, and the angle between the radii is called the **angle of the sector.**

Segment of a circle.—The figure bounded by any chord of a circle, and one of the arcs into which it divides the circumference, is called the *segment of a circle*, and the angle subtended by the chord of a segment at any point on its circumference is called the **angle in the segment,** and the segment is said to be **capable of the angle**

Similar segments.—Segments which are capable of equal angles are said to be *similar*

Solid.—A *solid* is that which has length, breadth, and thickness

Square.—A rectangle which has a pair of adjacent sides equal is called a *square*

Straight line.—A *straight line* is the shortest distance between two points

Supplementary, Supplement.—Two angles whose sum is equal to two right angles are called *supplementary angles*, and each angle is said to be the *supplement* of the other.

Surface.—A surface is that which has length and breadth, but no thickness

Symmetry, Symmetrical.—When a figure is such that it can be folded about a line so that the two parts exactly coincide, the figure is said to be *symmetrical* about the line, and the line is called an *axis of symmetry of the figure*

Transversal.—A straight line which cuts a number of other lines is called a *transversal*

Trapezium.—A quadrilateral, two of whose sides are parallel, is called a *trapezium*

Triangle.—A figure bounded by three straight lines is called a *triangle*

Vertex.—The point where the two arms of an angle meet is called its *vertex*, and any angular point of a rectilineal figure is a *vertex* of the figure.

ANSWERS

BOOK I

Proposition 1 3 Right angle 4 $45°$ 5 $137°$ 6 $90°$

3\. 1 Four equations added give the required result 2 OP, OQ bisectors of AOD, BOC, Fig Prop 3, draw OR bisector of COA

6 5 In each case there are two angles, one acute, the other obtuse If we take the acute angle in one case and the obtuse in the other, the angles are supplementary

7 1 $180°$ 3 $90°$, $30°$ 4 $90°$, $54°$, $36°$ 6 $45°$ 9 $60°$ 11. $36°$ $60°$ $84°$

8 1 $\left(1-\frac{2}{n}\right)180°$ 2 $108°$ 3 $120°$, $135°$, $144°$, $150°$ 4 Six. 5 Eight 6 Ten 7 Twelve 8 Eight

9 5 B, D lie on AX 8 In Δs ACA', BCB', $\angle ACA' = 60° + C = \angle BCB'$, sides about these angles are equal

10 7 See definition of kite

11 4 $75°$ 5 $72°$, $72°$, $36°$ 7 $60°$ 8 On one of the arms describe an equilateral triangle 10 $90°$, $45°$, $45°$ 11 Join P to the centre

12 6 Make an exact drawing on a large scale, measure various angles, and find reasons for their equality

16 5 $OB+OC>BC$, and two more inequalities.

17 3 Let it, if possible, cut in three points, join the points to the centre 4. Draw the perp from the vertex on the base 5 Triangle ABC, D mid pt of BC, produce AD to O, making $DO=DA$, join OC, and prove it equal to AB Apply Prop 16, Ex 1

18 2 From two pts on one of the parallels draw perps on the other, these perps are equal by Ex 1 3 (i) $ABCD$ quadrilateral, draw diagonal AC, apply Prop 10, (ii) apply Prop 13, (iii) apply Prop 4 and 10, (iv) Prop 10 8 Prop 9 9 Prop 13 12 Prove bisectors of opposite angles parallel, and of adjacent angles perpendicular

19 3 A, B, C angular pts.; D, E, F mid. pts. of opp sides; produce FE to G, making $EG=EF$; prove EAF, ECG congruent; hence $FBCG$ is a ▱ 4. Draw diagonal and apply Ex. 3 5 Follows from Ex 4 and Prop 18

MISCELLANEOUS QUESTIONS AND EXERCISES—I.

3 1 2; 4, 5; 11, 12, 15, 16 6. See *Exp Geom* 11. Prop 18. 12 When they are not in the same plane 13 Apply Prop 7, Cor 17 Six, viz. equilateral, isosceles, scalene, right-angled, obtuse-angled, acute-angled. 18 If $B-C$ be not greater than A, then it must be either equal or less What is the consequence in either case? 19 Def of a st. line 20 Six 21 Draw perps. to the sides 22 Prop. 9 23 Prop 9, join Q to mid pt of CP 24. Prop 16, Ex. 1. 25 AD bisector of BC, on AD produced take $DO=DA$, join CO Prove $CO=AB=AC$ 31 Draw perp from vertex on base 32 Draw perps from A on BC and CD Express BCD in terms of BAD 35 Apply last exercise 36 EC, BD intersect in O, prove $EC=BD$; prove OBC isosceles, then prove OED isosceles 37 Prove OBD isosceles. 38. First draw triangle with sides 5, 6, and 7 cm 42 Draw perp from A on BC 43 Eighteen 44 Twenty. 45 Join one mid pt with opp angular pts.; apply Prop 9 47 Start with drawing one diagonal, prove sum of sides greater than twice this diagonal Then draw the other diagonal. 49 Prop 16 53 Apply last two exercises 55 Square and equilateral triangle. 56 Thirty-six 57 Assume the contrary and apply Prop 17 60 Assume the contrary and see what results 61 Draw the bisector of the vertical angle AOB, the figure can be folded about this line. 62 Prop 9 63 This line coincides with bisector of vertical angle. 64 Line is parallel to DC; passes mid. pt of side AD of triangle ADC 65 Prove OEC, OBF congruent. 68 Draw parallel to AC through E, meeting BF in R 70 See Ex 65 71 In figure of Ex. 65 prove that a parallel to AB through O bisects both diagonals 73 $ABCD$ parallelogram, in which AB is greater than BC; from BA cut off $BP=BC$, and bisect CP in O, prove BO, CO bisectors of the angles at B and C Similarly construct the other bisectors 76 Line joining mid pts of sides of triangle is parallel to base and equal to half the base.

Proposition 20. 2 Place them on the same base 3 Place them on same base, and suppose, if possible, they have different altitudes:

4. Suppose, if possible, the bases are different, say AB, AC 5 This is only a particular case of the general theorem, and is very important The last three exercises follow easily from this. 6 If we take 7 cm as base, the height of the parallelogram is 2 5 cm, and area=17 5 sq cm 7 Each line is parallel to base 8 Divide base into five equal parts, and draw parallels through pts. of division

21 Cor 1 In figure of Prop 21 let $\parallel^{m}$ $ABPY$ and Δ ABC have same base and same altitude.

22 1, 2, 3, and 4 follow from Prop 21 Cor 2 9 Place the triangles so that supplementary angles are adjacent, you then have two triangles with equal bases and a common vertex This theorem is very important 10 Draw perps from vertical angles on common base, and prove them equal 11 and 12 follow from the fact that triangles with common vertex and equal bases are equal

23. 1 On OX, OY measure lengths a, b in succession and c, d; through pts of division draw parallels.

24. Take OX, OY $\perp$ to one another On OX measure the length $a+b$, and on OY the lengths a, b in succession Draw parallels.

26 6 Put $AB=x$, $BC=y$, $CD=z$, express the result to be proved in terms of x, y, z

27 Cor 1 This means that sq AD = ... BI 2 The hypotenuse and side of right angled triangle are given, construct it 9. Prop 2 10 Prop 4 11 If the triangle ABK be rotated through a right angle round the pt B, it will coincide with the triangle DBC, hence each side of one triangle is perpendicular to the corresponding side of the other in their original positions. This is a very useful method of proof 14 Follows from Prop 22, Ex 9 19 Join mid pt of longest side to opp angle

28 1 In the figure of this proposition with the given measurements find $x=1$, hence $p=2\sqrt{6}$ and area$=4\sqrt{6}$ sq cm 2 $p=1\cdot78''$, $\Delta=84$ sq in 3 $\sqrt{7}$, 1, 1 5 Follows from Ex 4 7 Apply Ex. 4, the perps are $2\cdot3''$ and $1\cdot9''$

29 1 $5\frac{1}{3}$ cm, $3\frac{2}{3}$ cm 3 $\sqrt{(39)}$ 8 $p=11\frac{3}{4}$ cm, segments are $81\frac{..}{..}$ cm and $\frac{...}{...}$ cm, $\Delta=546$ sq cm

MISCELLANEOUS QUESTIONS AND EXERCISES—II

5 Const similar to that of Ex 4 In each row there are m squares, and, since there are n rows, we have mn squares each of 1" side. 6 8 75 sq in 9 The sum=half the area of $ABCD$ 11 Apply

Ex. 10 13 p_1, p_2, p_3 perps, p altitude and a side of triangle; then $ap_1+ap_2-ap_3=2\Delta=ap$ Prop. 21, Cor 2 14. The sum$=2\Delta/a$, where Δ is the area of polygon and a its side. 16 Apply Ex. 15 17. Apply Prop 21, Cor 2 18 $\frac{1}{2}\sqrt{73}$. 19. 128 5 20 12 sq in 21. 5" 22 Prove that each diagonal bisects the other, by considering the join of the vertices of two equal triangles on opp sides of same base 23 Prop 29, Ex. 9. 24. The two trapeziums in the figure are congruent. 25 Prop. 29 Cor 27. 197 feet. 28 99 29 60°, 30°. 32. Apply Ex 31 33 Prove four triangles formed equal. 35. Prop 21, Cor. 2 36 6 3 cm 37 Bisect the angles between the diagonals, prove these bisectors the diagonals of a square. 38 Deduce from Prop 29. Cor, as a special case in which the vertex of the triangle is situated in its base or base produced 39. 136 sq cm. 41. $\frac{1}{2}(p_1-p_2)d$ 42. $\frac{1}{2}ad'$. 43 $\frac{1}{2}dd'$; draw perps. from the extremities of one diagonal on the other. 46. If not obtuse, then right or acute; what are the consequences? 49. Apply Ex. 47 51. The altitude of the triangle must be double that of the parallelogram. 54. In Prop 29, Ex. 9, $m_1^2 >$ or $< m_2^2$ according as? 55 Apply Ex. 54.

Proposition 30 1. The circle whose centre is the fixed point and radius the given distance. 2. Two lines parallel to the given line, at the given distance from it and on opposite sides of it 3 The right-bisector of the join of the two fixed points 4. Right-bisector of the base. 5 Circle whose centre is the given point and radius the given length 6 Two straight lines parallel to the given base, at the distance of the altitude from it, and on opposite sides of it.

31. 2 Let ABC be one of the triangles on the base BC. A parallel to BC drawn through A is the required locus Or, if Δ be the given area and $BC=a$, then the locus is a st. line parallel to BC, at a distance Δ/a from it. 3 A st. line parallel to the fixed line and bisecting the distance between the given pt. and fixed line. 4. A st. line parallel to the base at a distance equal to half the common altitude of the triangles. 5 A st. line parallel to the base at a distance equal to half the altitude of the parallelograms 6 A circle whose centre is at the middle point of the base, and radius equal to the given median. 7. Since the required pt. is equidistant from A and B its locus X is the right-bisector of AB; since it is equidistant from B and C its locus Y is the right-bisector of BC Therefore the required pt. is the pt of intersection of X and Y. 8 Draw the internal and external bisectors of the angles of the triangle formed.

by the lines. Prop 31 The intersections of these bisectors will give the required pts, which are four in number You had better make an accurate drawing 9 X is the right bisector of c, Prop 30, Ex 4 Y consists of two lines parallel to a at distances Δ/a on either side of it, Prop 31, Ex 2 The intersection of X and Y gives two points, hence two triangles can be constructed 10 X is the right-bisector of the join of given pts, Y is a circle. There are two solutions, one or none 11 X is the locus of Prop 31, and Y is a circle Draw figures for the various possible cases 12 X is the locus of Prop 31, Ex 2, but on both sides Y is a circle of radius m and centre at the mid pt of BC

32 3 All the chords have a common central perp 4 Follows from Ex 3 10 The bisector of parallel sides is an axis of symmetry

33 2 Let OA, OB, OC be the lines Since these are equal a circle with centre O will pass through A, B, C, and this is the only circle 3 If they intersect in a third point they must coincide 6 Draw the right-bisectors of any two chords 8 Diagonals are equal. 9 The line joining mid pts of parallel sides is an axis of symmetry

35 6 Draw the common right bisectors of the chords and apply Ex 5

36 2 and 3 Draw perps from the centre and apply Prop 27 4 Draw perps from the centre 6 12″ 7 17″ 8 224 feet 9 9″ 10 Draw perps from centre

37 1 Draw another 4 The perp at the given pt 6 Draw central perps to the chords 7 Join extremities of chord to the centre 8 Draw central perp on given line. 9 Draw through centre parallel and perp to given line 10 What angle does the radius to the point of contact make with the same diameter?

38 5 The centres are at the angular pts of a triangle whose sides are 3″, 4″, and 5″, see Ex 9 6 A circle of radius $R+r$ 7 A circle of radius $R-r$ 8 The line through the centre and the given pt 9 The centre are at the angular pts of a triangle, whose sides are (r_2+r_3), (r_3+r_1), (r_1+r_2), or (r_1+r_2), (r_3-r_1), (r_3-r_2) 10 The centre is on the line of centres one inch from the pt of contact of given circles 11 The perp through the given pt 12 The centre is at the intersection of the locus of Ex. 11 and the other line 13 The centre is the intersection of the locus of Ex 8 and the other circle There are several cases

40 5 The two points where the diameter, bisecting the base at right angles, meets the circumference 7 A, B, C three pts, draw any line BP and make the angle ACP=angle ABP, then P is the

required pt By varying the angle ABP we get any number of pts. in the circumference 8 The student will see further on that the proposition is general

42 1 The foot of the perp from the opp angle on the third side 2 Prop 41, Cor 3 Prop 42, Cor. 1. 6 Join two opp angular pts 7 Join extremities of remaining pair of sides cutting each other, let (a, a') and (b, b') be pairs of parallel sides Then angles between a, b and a', b' are equal, hence, by Prop 42, the remaining pair of sides and the drawn diagonals form isosceles triangles 8 Draw the diameter bisecting parallel sides at right angles, join centre to angular pts 9 Prop 41, Cor

43. 1 If it is not a tangent, then draw the tangent 3 Draw the tangent at pt of contact; the circles may touch internally or externally 5 Tangents at the extremities of a chord make equal angles with chord 6 Draw one circumcircle and draw the tangent at A, apply Ex 1 7 Draw one circle and draw the tangent at E, apply Ex 1

44. Cor 2 If OT is not a tangent, produce it to cut the circle again in T' Then

$$OP \cdot OQ = OT \cdot OT', \text{ but } OP \cdot OQ = OT^2,$$
$$\therefore OT \cdot OT' = OT^2, \text{ which is impossible}$$

1. Draw a secant and apply Cor 1 2 Apply Cor 1 to each circle 3 Converse of last 4. Follows from the last two The portions are the parts exterior to the circles 5 Let it cut a common tangent in O, the tangents from O are equal 6 A, B, C circles Let common chords of (B, C) and (C, A) meet in O, then the tangents from O to (A, B) are equal, therefore, by Ex 3, O lies on the common chord of (A, B) 7 Apply Ex 1 8 Let ORR' be the common chord, OPQ, $OP'Q'$ the secants, then $OP \cdot OQ = OR \cdot OR' = OP' \cdot OQ'$. Since $OP \cdot OQ = OP' \cdot OQ'$, therefore, by the converse of Prop 44, P, Q, P', Q' are concyclic 9 49 sq in 10 64 sq in 12 See next Ex 13 Draw a figure, drop a perp from opposite angular pt. on the side c, this perp x is half the common chord Also $cx = 2\Delta$, where Δ is area of triangle

MISCELLANEOUS QUESTIONS AND EXERCISES—III.

3 The centre 4 Consider sum of two sides in any triangle 5 Draw central perps on the line 6 Join the centre to the mid pts.

7 Drop central perps. on chords 11. (ii) It touches the locus of (i) Join the fixed pt to the centre, and prove the join of constant length 12 $\sqrt{(r^2-c^2)}$ 13 Drop central perps. on chords, and draw radii to pts of section 14 Perp to diameter which passes through the pt 16 Join pts of contact to centre 17 O centre of large circle, O' of small, let $OO'A$ be radius of large circle In order that the small circle may lie within the other its radius must be less than $O'A$ 18 Let OA be radius of one circle, then in order that the other circle may not cut it its radius must be less than $O'A$ 21 Prop 38, Ex 9 22 Drop central perp on chord 25 The diameter of the smaller circle is the greatest common chord 26 Drop central perps on chords, the locus is the circle described on the radius to the pt as diameter 27 The circle described on the join of the given pt and the centre as diameter 28 A portion of the circle described on the join of the given pt and centre as diameter 29 A concentric circle whose radius is the join of the centre and any extremity of a tangent 30 Let O be pt of intersection, drop perps from O on chords, prove these perps are in a st line. 31 Describe the circle circumscribing ABC, and let O be it's centre, then since A is given, the angle BOC is given, therefore BC subtends a constant angle at centre Hence length of BC is fixed and determined This is a very important theorem 32 On the radius to the point as diameter describe a circle which will pass through the feet of the perpendiculars Now apply Ex 31 33 Join the point of intersection to the centre 34 Prop 41 35 Prop 43, Ex 1 36 Let AB, CD be parallel sides of trap $ABCD$ Draw AL, BM perps on CD Prove $DM=CL$; hence prove $DL=MC$, hence $AD=BC$ AB, CD have same right-bisector about which the figure is symmetrical 37 $ABCD$ cyclic quadrilateral Let bisector of angle A meet circumference in P and DC in O Considering the triangles OAD, OCP, prove that PC bisects the exterior angle at C 38 Place the triangles on opposite sides of common base, the quadrilateral so formed is cyclic, Prop 42 39 Apply Ex 38 40 At the pt of contact two pts are common, if they meet in a third pt they must coincide altogether, Prop 33 42 Join the mid pt of ladder to the foot of the wall, the locus is a quadrant of a circle whose radius is half the length of the ladder 43 The chord joining the extremities of two radii at right angles 44. Chord joining the extremities of two radii whose angular interval is of 120° 45 $ABCD$ quadrilateral, produce (AB, DC) and (BC, AD) to meet in E, F Draw bisectors EO, FO cutting BC, DC in P, Q Put down the values of the

angles of the triangles EBC, FCD, obtain hence the angles OPC, OQC Finally find POQ 46 A circle on the hypotenuse as diameter. 47 A, B fixed pts, O mid pt. of AB, and P moving pt Then $PA^2+PB^2=2OP^2+2OA^2$ Prop 29, Cor Whence OP is constant Locus is a circle whose centre is O 48. Draw the line of centres 49 $ABCD$ quadrilateral inscribed in a circle, AC, BD intersect in O; OL perp to AB meets CD in P Then prove

$$\angle COP=\angle OBA=\angle PCO.$$

Proposition 45 5 See fig of Prop 19

46 2 (i) and (ii) Each side is a mean proportional between the hypotenuse and the segment adjacent to the side (iii) The perpendicular on the hypotenuse is a mean proportional between the segments of the hypotenuse (iv) The segments are in the duplicate ratio of the adjacent sides 3 Prop 42, Cor 1 4 Triangles formed by segments are equiangular 5 Prop 43 8 If $\frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'}$, then each fraction $=\frac{a+b+c}{a'+b'+c'}$ 9 $1\frac{1}{6}$ in 10 78 ft 11 $23\frac{97}{176}$ feet

48 5 $\frac{AP}{AC}=\frac{AC}{BQ}=\frac{BC}{BQ}$ Apply Prop 48 6 Prop 48 7 Prop 48 8 Draw AL perp to AB towards the same side of AB as C, and of any length, draw AM perp to AC towards the same side of AC as B, and make it half as long as AL Through L and M draw parallels to AB, AC meeting in O Then CO produced both ways is the required locus

49 1 Make the same construction as in the proposition, and apply Prop 45, part second 2 Let x=length of segment adj to 5; then $x/(7-x)=5/6$ The segments are $3\frac{2}{11}$ cm. and $3\frac{9}{11}$ cm 3 35 cm. and 42 cm 4 $\frac{ab}{b+c}$, $\frac{ac}{b+c}$, $\frac{bc}{c+a}$, $\frac{ab}{c+a}$, $\frac{ca}{a+b}$, $\frac{cb}{a+b}$

50 3 4 9 4. 441 400 5 3 4 6 400 sq yds 7 See Ex 10 9 Prove medians proportional to corresponding sides, by placing one triangle on the other 10 Let A, B be similar triangles on sides a, b, and C the similar triangle on hypotenuse c. Then $\frac{A}{a^2}=\frac{B}{b^2}=\frac{C}{c^2}$, $\therefore \frac{A+B}{a^2+b^2}=\frac{C}{c^2}$. Denominators are equal

MISCELLANEOUS QUESTIONS AND EXERCISES—IV.

4 $DE=3''$, $EC=1{\cdot}35''$ 5 $EF=\cdot 9''$, $FC=1{\cdot}8''$ 6 $2{\cdot}7''$ from A 7 $OK=\cdot 9''$ 8 Join ED, FD, $AEFD$ is a parallelogram 9 In similar triangles OGF, DGC, $OF=\frac{1}{2}DC$, ∴ $FG=\frac{1}{2}CG$ Similarly pt of intersection of BE, CF is a point of trisection of CF Hence medians pass through the same pt which is a pt. of trisection of each median 10 Find the ratio of the perps. from G and A on BC 11 Triangles have same altitude, use similar triangles 12 By rotating one of the triangles through a right angle, its sides can be made parallel to the sides of the other 13 L and M mid pts of AB, AC, prove triangles CAL, DAM similar 14 Apply Prop 49 and its converse 15 OAB, $O'A'B'$ are similar by const, $\frac{AB}{A'B'}=\frac{BO}{B'O'}$, but from def of similar figures $\frac{AB}{A'B'}=\frac{BC}{B'C'}$, ∴ $\frac{BO}{B'O'}=\frac{BC}{B'C'}$, and angles OBC, $O'B'C'$ are equal Hence OBC, $O'B'C'$ are similar In the same way all the other triangles can be proved similar

16.
$$\frac{AOB}{A'O'B'}=\frac{AB^2}{A'B'^2},\ \frac{OBC}{O'B'C'}=\frac{BC^2}{B'C'^2}=\frac{AB^2}{A'B'^2},$$
$$\therefore\ \frac{AOB}{A'O'B'}=\frac{OBC}{O'B'C'}=\frac{OCD}{O'C'D'}=\frac{ODA}{O'D'A'}=\frac{AB^2}{A'B'^2},$$
$$\therefore\ \frac{\text{Sum of numerators}}{\text{Sum of denominators}}=\frac{AB^2}{A'B'^2}.$$

17 16 25 18 4 1 19 Areas are proportional to the squares of the radii of their circumcircles; 3 4 20 1 2 23 The side of the third triangle is 2 cm 25 Prop 49 28 Prop 48 30 Besides proving the sides proportional, show also that the angles of the two polygons are equal

BOOK II

Proposition 52 4 The point must lie on the right bisector of AB, the right-bisector and AB can intersect in one point only

53 1 PQR is an equilateral triangle, $\angle QPR=60°$, $\angle RPL=30°$. 4 60°, 30° 5 A, B given pts, Y given line Draw X the locus of pts equidistant from A and B, Prop 30 Find pt of intersection of X and Y When X and Y are parallel the solution is impossible.

6 See Ex 3 7 The pt of intersection of the right bisectors of two adj sides

54 4 and 5 P, Q given pts. and Y given line Draw QL perp to Y and produce it to Q', cutting off $LQ'=QL$ Let PQ', produced if necessary, meet Y in O Then OP, OQ are required lines

55. 1 Let the circle whose centre is D cut the circle whose centre is A in CC'. Then CAC' is required angle 2 Produce CA to C'. 3 Draw AC' perp to AC lying on same side as D Then BAC' is required angle. 8 The dividing line makes the same angle with a side as the hypotenuse.

56 1 AB given line, P given pt At A make angles equal to given angle above and below AB. Through P draw parallels 2 OA, OB lines and P pt Construct a parallelogram of which P is the mid pt. of one diagonal The other diagonal is required line 3 Draw the bisectors of the angles between the given lines; draw perps. to bisectors through given pt You get two triangles 4 Place the given line with its extremities on parallel lines 5 The right-bisector of the join of given pts. is one of the diagonals of the rhombus. 6 OA, OB intersecting lines; let third line in question intersect them in A, B From AB or BA cut off a length equal to fourth line There are therefore two solutions.

59 2 Draw parallels to opp sides through the angular pts 3 Join mid pts. of sides

62 2 First construct a triangle on base 7 cm whose sides are half the lengths of the diagonals Then complete the parallelogram 3 Construct a triangle with the given lengths

65 2 In the figure of the prop draw the right-bisector of MC meeting ADN This is one of the semi diagonals of the rhombus

MISCELLANEOUS EXERCISES—V.

4. Let APB be the arc and AB its chord With A and B as centres and any equal radii describe arcs intersecting in L and M. Let LM cut the arc in P, then prove that arc AP=arc BP. Note that the construction is the same as for bisecting a line AB See Prop 52 and Cor 20 In the figure of Ex 19, from CA cut off $CP'=a$ The pt C lies on the right bisector of BP' 24. ABC triangle in which c, C and $a+b$ are given *Analysis*—Produce AC, cutting off $CP=a$ In triangle APB two sides and an angle opp to one of them is given The triangle can be constructed Prop 60. Also notice that C lies on the right-bisector of BP. There may be two solutions.

25 From CA cut off CP' equal to CB C lies on the right-bisector of BP' 30 $ABCD$ square, from DB cut off $DP=DA$ In the triangle APB the side BP and the angles are known 34 In the figure of Ex 32, from AC produced cut off $CD'=CB$ The angle DBD' is a right angle. Hence two sides and one angle of the triangle ABD' are given Prop 60 36 The three angles are given, hence Prop 59 applies here 38 The bisector of the right angle is a diagonal of the square 40 The bisector of an angle is the diagonal of the rhombus 41 Triangle ABC, through A draw AP parallel to BC and cut it off equal to the altitude AL. Join P to the base angle B which is on the other side of AL If PB cut AC in S, then S is an angular point of the square 42 Consider the whole figure as a triangle and a trapezium Then $\frac{1}{2}(p-s)s+\frac{1}{2}(a+s)s=\frac{1}{2}pa$ 43 and 44 The locus of the vertex is a straight line parallel to the base at the distance of the altitude from it 45 1 2″ 48 $ABCD$ quadrilateral. Bisect BD in M, and draw MR parallel to AC meeting CD in R. Then AR will bisect the quadrilateral Join AM, CM, and prove that quadrilateral $ABCM$ is half of quadrilateral $ABCD$. The student ought to supply the analysis for this construction 51 BC base of the triangle ABC Describe a semicircle on AB and let the right-bisector of AB meet the semicircle in Q Let the circle whose centre is A and radius AQ meet AB in P A parallel to BC drawn through P will bisect the triangle 52 BC base of ABC Draw AL at right angles to BA meeting BC produced in L. Bisect BC in D, and with DL as diameter describe a circle. Let BT be the tangent from B to this circle From BA cut off $BP=BT$ Through P draw PN perp to BC Then PN will bisect the triangle 53 Draw a line through the given pt and the pt of intersection of the diagonals 54. The line through the centres of the two parallelograms 56 First construct a right-angled triangle having given side for hypotenuse and given perp for side 57 *Analysis* —Suppose ABC triangle Produce BA to D, making $AD=AC$ Draw DE parallel to AC meeting BC in E Join DC BD is given, also angles of triangle BED DC is bisector of angle at D 58 *Analysis* —Similar to last. 59 *Analysis* —ABC triangle, AD median Produce AD to E, taking $DE=DA$ Join BE Sides of triangle ABE are given 60 On a diagonal of the square as base describe an equilateral triangle. The bisector of the base angle of this triangle lies in the direction of a side of the triangle required 61 See Ex 30 62 Prop 60 63 The right angle lies on a certain semicircle 64 Let x=length of required line, then $x-a=b-x$, $x=\frac{1}{2}(a+b)$ 65 (1) Join EB and prove

that BE is the bisector of angle B, hence the construction (ii) and (iii) Draw ED parallel to AB meeting BC in D. In triangle EDC, $ED \pm EC$ is given and the angles are given. Hence the triangle can be constructed in both cases Exs. 57 and 58 66 The bisector of any angle is the axis of the corresponding kite, hence there are three solutions 67 Divide the polygon into triangles by lines drawn from an angular pt. 68 Triangle ABC, draw a parallel to BC at a distance of half the altitude of the triangle. With centre B and radius BC describe an arc cutting the parallel in D Then BCD are angular points of rhombus 69 Reduce the quadrilateral to a triangle, Prop 64 Now apply Ex 46 71 Intersection of medians 72 Bisect the sides of the square, and hence obtain the diagram given Apply Prop 50 73 The altitude of the equilateral triangle on a side of the given square is equal to a side of the required square 74 Through the pt. of intersection of the given lines, draw a line equal to one line and parallel to another, then complete a parallelogram 75 ABC given triangle; bisect BC in D, with centre D and radius equal to $\frac{1}{2}(AB+CA)$ describe a circle cutting the parallel to BC, drawn through A, in P. Then DP, DC are sides of the parallelogram

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Proposition 66 6 Let TT' cut OP in L, then TL, half the chord of contact, is the perp from the right angle on hypotenuse, $\therefore$ $OT^2 = OL \cdot OP$ and $TL^2 = OL \cdot PL$. 10 Apply Ex 2 11 Apply Ex 2 12. Apply Ex 1 13 Apply Ex 1

67 2 The centre of each circle must lie on the bisector of the angle between two tangents to the circle 4 As the circles approach one another, the common tangents become shorter and shorter (see Ex 5), while the angle between them becomes more and more acute, until at the pt of contact their *lengths* vanish and their *directions* become the same 5. From the right-angled triangle ABC in each case 6 In the first formula of Ex 5 put $c = R + r$

68 3 $4\frac{1}{8}''$ 4 $1\frac{9}{20}''$ 5 $1\frac{1}{4}''$ 6 Use the formulas of Ex. 2, or else the construction suggested by Ex 8 below 7 Follows from Ex 6 8 Prop 39 9 Prop 40 and Ex 8 11 Place the triangles base to base and they will form a cyclic quadrilateral Prop. 42

69 2 Prop. 66, Exercises 5 Use Ex 4, putting $a' = na$. 7 Prop 36 8. 1 2". 9 2 4" 10 1 4", 1 2", 1.2", 1.7"; 1.7", 1.4".

70 8 Apply Exs 5 and 4 9 We have

$$\Delta ABI_1 + \Delta ACI_1 - \Delta BCI_1 = \Delta ABC;$$

$$\therefore \tfrac{1}{2}cr_1 + \tfrac{1}{2}br_1 - \tfrac{1}{2}ar_1 = \Delta,$$

i.e.
$$\tfrac{1}{2}(c+b-a)r_1 = \Delta, \quad \therefore r_1 = \frac{\Delta}{s-a}.$$

72. 4 Let BC, $B'C'$ be two corresponding sides and O centre. $\angle BOC = \angle A$, $\angle B'OC' = \angle A'$ 6 *Analysis* —Suppose the triangle described as required Join the centre to the pts of contact of sides. Prove that the angles between the radii are the supplements of the angles of the triangle Hence the construction

73 1 Describe a semicircle See why the given construction fails in this case 5 On two sides of the triangle, and lying inwards, describe segments capable of 120° Let O be their pt of intersection Find the angles subtended by the sides at O 7 The locus of the vertex is the segment of a circle capable of the given vertical angle The locus of the vertex is also a line parallel to the base at a distance equal to the given altitude The intersections of these two loci determine the positions of the vertex In general there are two solutions, but there may be only one, or none 8 The altitude can be found 12 Use intersection of two loci, notice that the locus of the other extremity of the median is a circle 14 Make an analysis similar to that given in the text, Ex 6

74. 5, 6, and 7 Draw lines and divide the figures into a number of equal triangles 8 Prop 28

75 10 First construct a rectangle equal to the parallelogram and standing on the same base as the parallelogram 12 First construct the equivalent rectangle, Prop 65 13 Construct an equivalent rectangle on 2 5″ as base. 15 OX, OY given lines, and P given pt Draw PA parallel to OX meeting OY in A Divide OA into four equal parts, along AY step off AR to contain three of these parts Produce RP to meet OX in Q Then RPQ is the required line 16 As in Ex 15, draw PA parallel to OX Divide AO into four equal parts, along AO step off AR to contain one of these parts 17 Join the mid pt of the base of the triangle to the mid pts. of the other sides You now have a parallelogram of the species required, but containing only half the area required Increase the sides of the parallelogram in the ratio of 1 $\sqrt{2}$, the ratio of a side of a square to its diagonal

76 4. Find a mean proportional to $\sqrt{xy}$ and $\sqrt{zu}$ 5 Find a mean proportional to the lines constructed in Ex 3

77 6 In the figure of Prop 77, AB and BL are given, hence triangle ABL is given LP drawn at right angles to AL meets AB produced in P This gives BP *Otherwise* use the method of Ex. 8, Prop 75 8 The lesser side of the right-angled triangle is equal to a side of the square which is equivalent to the polygon 9 The altitude of the isosceles triangle is equal to a side of the square which has the same area as the polygon

MISCELLANEOUS EXERCISES—VI.

1 2″ **2** 3 5″. **3** $1\frac{7}{8}''$ **4** Central distance of the point $=\sqrt{2}\times$ radius. **5** The radii to the pts of contact contain the supplement of the given angle, a concentric circle **6** Central distance of the point $=\sqrt{5}\times$ radius **7** Direct tangents 4 cm each, oblique, zero Note that the oblique tangent has *direction* but no *magnitude* **8** Prop 44, Exs **9** 12 cm **10** 1 6 cm **12** Mid pt of hypotenuse is centre **14** O circumcentre of ABC, $\angle AOC=2B$; $\angle OAC=90^\circ-B$, similarly, $\angle OAB=90^\circ-C$. If AL be altitude, then $\angle LAC=90^\circ-C$, $\angle LAB=90^\circ-B$ Therefore $OAB-LAB=B-C$. The other results given here are also useful **15** The circumcentre is equidistant from two sides **19** The two loci on which the centre lies are (i) the bisectors of the angles between the given lines; and (ii) the right-bisector of the join of the given pts **20** The loci on which the centre lies are (i) the radius through the given pt on the circle, and (ii) the right-bisector of the join of the two given pts **21** The loci on which the vertex lies are (i) the segment of a circle, and (ii) the right-bisector of the base **23** The centre lies on (i) the right-bisector of BC, and also on (ii) a circle, centre A and radius 2 4 cm **24** The locus of the centre is (i) the bisectors of the internal and external angles at A, and (ii) a pair of straight lines parallel to BC, on either side of BC, and at distances of 1 2″ from BC **25** Sides subtend equal angles at centre, two radii and a side form an isosceles triangle whose base angles are given, each angle of figure is double of one of these base angles **27** Consider a five sided figure with sides a, b, c, d, e By 26 we have $a=c=e=b=d$ **28** Let PQ, QR, RS be three consecutive sides, A, B, C their pts of contact. Join OA, OB, OC, and prove that OQR is an isosceles triangle in which the height OB and vertical angle QOR are constant **29** Consider the five sided figure $ABCDE$ described about the circle whose centre is O Let P and Q be the pts of contact of the sides BC, DE Angle $OAB=\angle OAE$ Triangles OAE, OAB congruent Hence $\angle OEA=\angle OBA$ and $OE=OB$ Also in rt-angled triangles OQE, OPB, $OB=OE$, $OP=OQ$, $\therefore$ $\angle OBP=\angle OEQ$ Therefore whole angle AEQ=whole angle ABP Similarly, other alternate angles can be proved equal Hence $\angle A=\angle C=\angle E=\angle B=\angle D$ Therefore the figure is equiangular in the case of a pentagon. Similarly, it is equiangular in the case of any other figure with an odd number of sides **34** Draw the circumcircle of ABC, bisect the arcs BC, CA, AB in D, E, F. Join

EF, FD, DE, then these lines will cut out a hexagon from the triangle. 38 Draw the circumcircle of the square and proceed as in Ex. 34. 39 Proceed as in Exs. 30 and 35 Divide the semicircle into six equal parts by trisecting right angles. Let C be the pt of division nearest P Then BC is a side of the dodecagon 44 Let $I_1I_2I_3$ be the excentre triangle Draw I_1A, I_2B, I_3C perps on the opposite sides. Then ABC is the required triangle. Prop 70, Ex 8 45 I_2, I_3, I are given Draw perps. from the angular pts on the opposite sides of the triangle I_2I_3I The feet of these perps. are the angular pts of the required triangle Prop 70, Ex 8. 46 Since A is bisected by AM, arc MB=arc MC $\angle BMC=180° - A$, $\angle MCB=\frac{A}{2}$, $\angle MCI=\frac{A}{2}+\frac{C}{2}=\angle CIM$, $MC=MI$ Similarly, $MB=MI$ 47 The circle ABC is given, and since $MB=MC$ the pt M is given in position and MB is given Since $MI=MB$ the locus of I is a circle whose centre is M and radius MB 48 The $\angle ICI_1$ is a right angle, $MI_1=MI$ Hence the locus of I_1 is the circle whose centre is M and radius MB 49 In the figure of Prop. 69 suppose a circle has been drawn which touches BA, BC, and the circle DEF Since this circle touches BA, BC, its centre O must lie on BI, the bisector of the angle between BA and BC Also, since this circle touches DEF, the pt. of contact must be the pt. P where DEF cuts OI Through P draw QPR perp to BI Then the required circle is the incircle of the triangle QBR The construction follows from this analysis There are several solutions, but what we usually require is the circle in the angular space between B and the incircle 51 Suppose the rhombus constructed as required, then show that two of the radii to the pts of contact of the inscribed circle contain an angle equal to an angle of the rhombus 52 Use the formulæ for r and r_1 53 $AI^2=r^2+(s-a)^2=\frac{\Delta^2}{s^2}+(s-a)^2$, and $AI_1^2=\frac{\Delta^2}{(s-a)^2}+s^2$ Multiply together and reduce. 54. OAB sector, centre O, suppose that a circle has been inscribed Draw a tangent at the pt. of contact of incircle and arc of sector Show that the incircle is also the incircle of the triangle formed by the tangent and two radii produced 55 See figure of Ex 46 The analysis of the problem is in this figure Draw a circle with the given circumradius Cut off the segment BAC capable of the given vertical angle. Bisect the arc BC in M The locus of I is the circle whose centre is M and radius MB Describe this locus The locus of I again is a parallel to BC drawn at the distance of the given inradius Draw this parallel. Let the circle

locus of I and the line locus intersect in I, I' Then I or I' may be taken as the incentre Join MI and produce it to meet the circumcircle in A Then ABC is the required triangle 56 See figure of Ex 46 On the given base BC describe the segment BAC capable of the given vertical angle Bisect BC in M, produce MI to meet the segment BAC in A Then ABC is the required triangle 57 The construction follows from the figure of Ex 46, as in the last exercise 58 Inscribe a circle in the given triangle The pts of contact are the pts where the required circles touch each other 59 Draw the medians AD, BE, CF of the equilateral triangle ABC meeting in G Inscribe circles in the quadrilaterals $GDBF$, $GDCE$, $GEAF$. 60 Inscribe circles in the triangles GBC, GCA, GAB 61 $r=\frac{\Delta}{s}$, $r_1=\frac{\Delta}{s-a}$ 62 On the given base BC describe a segment capable of the given vertical angle. With centre B and radius equal to the given perpendicular describe a circle Draw a tangent to this circle from C and let it meet the segment in A Then ABC is the required triangle. 63 See figure of Ex 46 Here M is given; join M to the given pt and produce the join to meet the circumcircle in A 64 A concentric circle whose radius $=\sqrt{2}\times$(radius of given circle) 66 Proceed as in the last exercise. We have $a^2=2R^2$ and $c^2=2r^2$ Thus r is a fourth proportional to a, c, and R. The student will notice that the method is general, and will apply equally well to other regular figures 68 Produce a radius so that the part produced is equal to the radius This determines one centre For the others draw radii at intervals of 60°. 69 Let R be the circumradius of the equilateral triangle formed by joining the centres of the required circles, and r the radius of the given circle, then an analysis of the figure will show that R $r=$(a side of an equilateral triangle) (difference between side and altitude) Hence R can be found 70 Since the sides are tangents to a concentric circle, they are chords equidistant from the centre and are therefore all equal. Again, an equilateral inscribed polygon is equiangular also.

BOOK III

SECTION I

Proposition II 4. Note When is the area of a triangle positive, and when negative? Walk round the contour of a triangle in the

direction indicated by the letters at the angular points If the triangle is to your left hand, its area is positive, if to your right hand, the area is negative Thus, if the area ABC is positive, the area ACB is negative. In this exercise let p be the perp from O on the line BC, then $\Delta OBC+\Delta OCA+\Delta OAB=\frac{1}{2}p(BC+CA+AB)=0$, by Ex 1, or by considering the signs of the areas 5 Proceed as in last Ex, using Euler's Theorem 6 Put $OA=a$, $OB=b$, $OM=\frac{1}{2}(a+b)$, $AM=\frac{1}{2}(b-a)$, $MB=\frac{1}{2}(b-a)$, $AB=b-a$ 7 and 8 See solution of Ex 9 9 Draw PO perp on ABC, and take O as origin Let $PO=p$, $OA=a$, $OB=b$, $OC=c$ Then $AP^2 \cdot BC=(OA^2+OP^2)BC=(a^2+p^2)(c-b)$, $\Sigma(AP^2 \cdot BC)=p^2\Sigma(c-b)+\Sigma a^2(c-b)=\Sigma a^2(c-b)=-(c-b)(a-c)(b-a)=-BC \cdot CA \cdot AB$ 10 This is a particular case of the last when P and O coincide Prove it independently, taking P as origin

III 2 Let FG, DP meet in O Since DF is divided in medial section at A, by similar triangles DO is divided similarly at P $DP \cdot PO=AP \cdot PB$ Here BO is parallel to AD 4, 5, 6, 7, and 10 Let $AB=a$, $AP=x$, then $PB=a-x$ For medial section we have $x^2=a(a-x)$ This relation is to be used in each case Thus in (5) we have $(AB+PB)^2=(2a-x)^2=4a^2-4ax+x^2=4a(a-x)+x^2$ But $a-x=\frac{x^2}{a}$, $(AB+PB)^2=4a\frac{x^2}{a}+x^2=5x^2=5AP^2$ 8 In the Δs FAB, APD a rotation through one right angle will bring one of them into coincidence with the other, hence corresponding sides are perpendicular 9 The triangles CLP, LDF have one angle of the one equal to one angle of the other In order that they may be similar we must have $\frac{CL}{LP}=\frac{DL}{DF}$, i e $\frac{a-x}{a}=\frac{x}{a+x}$, i e $x^2=a(a-x)$, which is the case

IV 2 Join alternate angular pts. of regular decagon 6 $\angle BAC=36°$, $\angle ABC=72°=\frac{4}{5}$ rt $\angle$ 7 (One angle of ΔABC)−(angle of equilateral triangle)$=\frac{2}{15}$ rt $\angle$ 9 On the given base describe a triangle similar to ABC 10 (ii) The angle subtended by CP at the circumference$=36°$, at the centre$=72°$ 11 Triangle similar to APC 12 On the given base BC describe a triangle ABC having each of the angles at the base double of the vertical angle Draw the circumcircle of ABC Let the bisectors of $\angle$s B, C meet the circumcircle in D, E Then $ADCBE$ is a regular pentagon 14. (i) BC is a tangent and CA a chord through the pt of contact, $\angle ACB=\angle ADC$ (ii) $\angle CAD=\angle CAB=\angle PCA$ (iii) $\angle CPD=\angle PAC=\angle ACP=\angle PCD$ (iv) Angle subtended by PD at C equals angle subtended by AC at P 15 Construct the figure of this prop Draw CL perp to PB and produce it to meet the circle DCB

in Q Then CQ is a side of a regular pentagon inscribed in the circle. Also, $\because$ $CP=CB$, $\therefore$ $PL=LB$ Prove that

$$AB^2+AP^2=2BL^2+2AL^2 \quad . \quad . \quad . \quad (1)$$

$$2AC^2+2BC^2=2AL^2+2BL^2+4CL^2=2AL^2+2BL^2+CQ^2 \quad (2)$$

Substitute in (2) from (1), then $2AC^2+2BC^2=AB^2+AP^2+CQ^2$. But $AB=AC$, and $BC=AP$, $\therefore$ $AB^2+AP^2=CQ^2$, where AB, AP, and CQ are the sides of a regular hexagon, decagon, and pentagon inscribed in the circle

Harmonic Section —1, 2, 3, and 4 are all proved by using the defining relation $AX/XB=AY/BY$ and taking one of the pts as origin 5 Bk I, Prop 49

Proposition V 1 The required pt is determined by the intersection of two Apollonian loci 2 Is a particular case of Ex 1

VI 2 to 5 In each case apply the criterion Notice that the segments of the side BC made by the point of contact of the escribed circle are $s-c$ and $s-b$

VIII 1 The product of each set of alternate segments $=\frac{1}{2}a\ \frac{1}{2}b\ \frac{1}{2}c$ 2 If AA' be a bisector, then $A'B/CA'=c/b$ 3 If AA', BB' be altitudes, then from similar triangles $CA'/B'C=b/a$, and two similar results 4 If A' be pt of contact, then $A'B=s-b$, $CA'=s-c$ 5 $OA'/AA'=\Delta OBC/\Delta ABC$ 6 $CA'=s-b$, $A'B=s-c$, similarly for the other sides 7 The alternate segments are $(\frac{2}{3}a, \frac{3}{4}b, \frac{1}{3}c)$ and $(\frac{1}{3}a, \frac{1}{4}b, \frac{2}{3}c)$ 8 General case of Ex 7 9 Let A', B', C' be the pts of contact with the sides BC, CA, AB, the segments of BC are as in Ex 6 The segments of AB are $C'A=s$, $BC'=s-c$, similarly for AC

IX 1 If A', B', C' be the points of meeting, then the ratios $A'B/CA'$, $B'C/AB'$, $C'A/CB'=$ the ratios AB/CA, BC/AB, CA/CB. Bk. I, Prop 49 2 Apply Bk. I., Prop 49

SECTION II

EXERCISES ON MAXIMA AND MINIMA

1 Let the parts AB, BC, CD . remain fixed, while the last two, HK, KL, vary Then HK^2+KL^2 can be made less by putting $HK=KL$, Prop VIII 2 As in the last, $HK\cdot KL$ can be made greater by putting $HK=KL$ 3 They lie along the line of centres 4 Draw APB making equal angles with OX, OY A circle can be described to touch OA, OB at A and B Draw another line $A'PB'$ terminated by OX, OY Then $A'P\cdot PB'>AP\cdot PB$ 5 Prop VII

6 Join mid pts of sides and apply Prop VIII 7 and 8 See Prop. I 9 Use Prop I, noticing that area of quad = area of triangle, whose sides are diags of quad, and included angle equal to that between diags 10 Let M be mid pt of AB, then $PA^2+PB^2=2AM^2+2PM^2$ Thus MP must be made a min Hence M and P are collinear with centre 11 The centre of the square See Prop VIII 12 Tangents at the pts. where the line of centres cuts the inner 13 Equilateral triangle, keep one side fixed and let the other two vary 14 Makes equal angles with the given tangents For max area mid pt of minor arc, for min area mid pt of major arc. Use the general method as in Prop II 15 Same as last Both tangents are minima. 16 Follows from 14. Note that $2\times$area=perimeter$\times$radius 18 It is parallel to line of centres See the construction for drawing a direct common tangent 19 ABC, $A'BC$, two isosceles triangles, such that AA' is indefinitely small For maximum vertical angle $\angle BAC=\angle BA'C$, B, A, A', C are concyclic. The tangent AA' is parallel to BC 20 $PQ/AB=CP/CB$, $PR/CA=BP/CB$, $\therefore \dfrac{PQ\ PR}{AB\ AC}=\dfrac{CP\ BP}{BC^2}$, i e $PQ\ PR=\dfrac{AB\ AC}{BC^2}\ CP.BP$ Now the angle of the parallelogram is constant, hence the area varies as $PQ\ QR$, area is a max when $CP\ BP$ is a max, *i e* when P is the mid pt of BC

SECTION III

Proposition I 1 $GB+GC>BC$ and two other equations 3 Join CM and prove $BCMF$ a parallelogram, hence $CMAF$ a parallelogram, also $BEMD$ a parallelogram 4 In the figure of Ex 3 prove E the centroid of MAD Then MAD being given, E can be found, and hence ACB can be constructed Notice that *the medians of the triangle MAD are each three-fourths of the corresponding sides of ABC* 7 $m_1^2=\dfrac{2b^2+2c^2-a^2}{4}$ and two similar expressions for the medians Also $AG^2=\frac{4}{9}m_1^2$ 8 If G be anywhere on the circumference $AG^2+BG^2=2AF^2+2GF^2$ Now AF is constant, hence we have to make GF a min This will be the case when FG passes through the centre 9 Let G be the required point, and suppose that CG is kept constant Then the locus of G is a circle By the last Ex AG^2+BG^2 is a min when G lies on CF Similarly, G lies on each of the other medians Hence G is the centroid of ABC 11 The altitude is

given; the locus is a st. line parallel to the base at a distance of one-third of the altitude from the base. 12 BC given base. Cut off $DB'=\frac{1}{3}DB$ and $DC'=\frac{1}{3}DC$. Join $B'G$, $C'G$. Then $B'G$, $C'G$ are parallel to BA, CA Therefore triangle $B'GC'$ is equiangular with ABC Locus of G is a segment on $B'C'$ similar to the segment on B', 13 Draw DK, GR perps to the locus of the vertex. Then DK is fixed, and from similar triangles $GR . DK=2 . 3$ Therefore locus of G is a st. line parallel to the locus of the vertex at two-thirds of the distance of D from it

II 1. The triangle $A'B'C'$ is of twice the linear dimensions of ABC. The right-bisectors of the sides of $A'B'C'$ meet in its circumcentre, but these lines are also the perpendiculars of ABC PA and OD are corresponding lines in the two triangles, hence $PA=2OD$ 2. Turn the triangle PBC round BC until it falls on the opposite side of BC You have now a cyclic quadrilateral $ABPC$ 4. $\angle PLM=\angle PCM$ $=90°-A$ Similarly, $\angle PLN=90°-A$. Hence $MLN=180°-2A$. 5 See Ex. 4. 6 Keeping MN constant $ML+NL$ is a min. when ML and NL make equal angles with BC. Sec. II., Prop XI. 7. LP bisects the angle MLN. 8 BL perp to PL is the bisector of the supplement of MLN. 9 OD is constant. 10 The perp and circumradius are equally inclined to sides.

III 1 and 2 They have the same orthocentric triangle. 3 Each =circumradius 4. EF, $\beta\gamma$ each parallel to BC; βF, $E\gamma$ each parallel to AP; also AP perp to BC *This theorem and the next are important, as they prove the existence of the nine-point circle.* 6 and 7 $GP=2OG$, and μ is the mid. pt. of OP 9 Here O, A, P are given in position, and also R $\therefore$ O and P are given, μ is known; and since $\frac{1}{2}R$ is given, the nine-pt. circle is given. Produce AP to meet the N P circle in L With O as centre and rad. $=\frac{1}{2}AP$ describe a circle. Through L draw a tangent to this circle and let the tangent intersect the given circumcircle in B, C Then ABC is the required triangle. See figure of Ex. 2. 10. We have proved before that I is the orthocentre of $I_1I_2I_3$, hence ABC is the N.P. circle of $I_1I_2I_3$, $\therefore$ it bisects the sides of this triangle and also II_1, II_2, II_3.

IV. 1. The axis of reflexion is an axis of symmetry of the whole figure. Hence by folding, each pt. of the original figure will fall on the corresponding pt. of the image. 3 In the figure of this proposition suppose the arc $BP'C$ is turned round the base BC until it falls on the other side of BC Then P' will fall on P. Hence the locus of P is the image of the arc $BP'C$ in the line BC. Notice that when BC and $\angle A$ are given, the circumcircle and consequently the arc $BP'C$ are fixed. Therefore

the required locus is found by making an equal circle pass through B, C 4. Follows from Ex 3 5 By Ex 4 the orthocentre of ΔPAB lies on circle BPC and also on circle APC Hence P is orthocentre of APB, in the same way P is orthocentre of BPC and of CPA Therefore P must be the orthocentre of ABC See Prop III, Ex 1

VI 1 The segments of the base are $\frac{ca}{b+c}$, $\frac{ba}{b+c}$ 2 The segments of the base are $\frac{ca}{c-b}$, $\frac{ba}{c-b}$ 3 In the second figure, if the bisector of the internal angle meet the arc in T_0, then TT_0 is a diameter 4. Δs $TD'C$, TAC are similar 5 Δs $TD'C$, TAC are similar. 6 Δs ABD', ATC are similar 7 Δs ABD'', ATC are similar

VII 3 and 4 have been proved before 5 $\angle BD'Q=2\angle BCQ$, $QD'=CD'$ 6 In the first figure BQ is given, DT is given The locus is a circle whose centre is the mid pt of base and radius half the difference of the sides 7 From the second figure, the locus is a circle whose centre is the mid pt of the base, and radius half the sum of the sides 8 AL is the altitude

VIII 1 Follows from the proposition by changing the sign of n 2 Put $m=1=n$ 3 Here $4BQ=3CQ$ Hence $AQ=\frac{1}{2}(135/7)^{\frac{1}{2}}$

IX 1 ABC triangle and D pt on circumcircle Apply this theorem to the quadrilateral $ABCD$ 2 ABC and CQD are similar 3 Area$=\frac{1}{2}AC \cdot BD$ in this case 4 From A, C draw perps on BD The sum of these perps $=\frac{1}{2}AC$ Also area$=\frac{1}{2}BD\times$sum of perps 5 Let p_1 and p_2 be perps from B, D on AC Then $p_1/p_2=BR/DR$ Again, $AB \cdot BC=p_1\times$diam, and $AD \cdot CD=p_2\times$diam.

SECTION IV

Proposition I 5 $\sqrt{2}$ inches

II 7 Find O the radical centre, draw a tangent from O to one of the circles, with O as centre and the tangent as radius describe the orthogonal circle. 8 Besides the angular pts A, B, C these circles intersect in L, M, N, the feet of the altitudes Hence AL, BM, CN are the radical axes 9 Altitudes are the radical axes. 10 Let the radical axis cut a common tangent TT' in P Then since P is on the rad axis, $PT=PT'$

III 4. See Ex 9 12 Let p, q be the lines intersecting in T, and P, Q their poles The polar of P passes through T, the polar of T passes through P Similarly, the polar of T passes through Q

SECTION V

Proposition II. 2 Suppose the square described as required, describe another square on a side of the hexagon internally Then the homothetic centre is the intersection of two sides of the hexagon 3 Bisect a side of the triangle in O, on either side of O cut off from the side two equal convenient lengths OA, OB On AB describe a regular octagon Then O is the homothetic centre of this octagon and the required octagon 4 Let O be the mid pt of base On the base describe a regular hexagon Then O is the homothetic centre of this hexagon and the required hexagon 5 On the other diagonal describe a regular pentagon Then the homothetic centre of this pentagon and the required pentagon is an angular pt of the square. 8 ABC triangle, AL altitude On AL describe a square lying towards C Then B is the homothetic centre of this square and the required square 9 ABC triangle, and PQR the triangle to which required triangle is similar Draw any line XY parallel to the given line, and meeting AB, BC in X, Y On the side of XY remote from B describe a triangle XYZ equiangular to PQR Let BZ produced meet CA in M Draw ML, MN parallel to ZY, ZX Then LMN is the required triangle Notice that B is the homothetic centre of XYZ, LMN. 10 Is a particular case of Ex 9 11 Particular case of Ex. 9 12 BC longest side of ABC On the side of BC remote from A describe a rhombus having an angle of 120° Then A is the homothetic centre of this rhombus and the required rhombus

IV 3 Place p on P, and make ab parallel to AB 4 Let P, p be their centres, place an indefinitely small chord AB in circle P so that ΔAPB and sector APB may be ultimately equal In circle p make $\angle apb = \angle APB$ Then Δs APB, apb are similar Again, make $\angle$s BPC, bpc indefinitely small and equal, then Δs BPC, bpc are similar, and so on, until the two circles are divided into the same number of very small similar triangles Thus by the converse of Prop III the two circles are similar 6. The corresponding sides of the small similar triangles, into which the two circles can be decomposed, are in the ratio of their radii; hence by Ex 5 the circumferences are in the ratio of the radii 7 Apply Prop IV after decomposing the circles into small triangles

VI. 5 The circumcentre O is a fixed pt, and we have proved before that the locus of the orthocentre P is a circle Now $OG/OP = 1/3$; $\therefore$ locus of G is a circle, Ex. 4. Again $O\mu/OP=\frac{1}{2}$, $\therefore$ locus of μ is a circle.

MISCELLANEOUS EXAMPLES—VII.

1. Let tangent at C cut PQ in T, PCQ is right angled triangle. 2 $AF=AL=S$, each triangle$=\frac{1}{2}rs$. 3 In any triangle $bc=2R\cdot p_1$ where p_1 is the alt of A Join the pt to angular pts. of quad. 4. Prove CPQ, BPD equiangular, notice that arc BD=arc BC. 5 Let BP, QR intersect in O, $OQ=OB=OR$, ∴ $\angle QBR$ is a rt $\angle$ Also $BP^2=AB \cdot BC$, and $QR^2=AB \cdot BC$ Prove $QBRP$ a rectangle, $OB=\frac{1}{2}QR=\frac{1}{2}BP$ 6 Let AP be the fixed circle Let the common chord intersect AB in O Then $OA=OB$, ∴ O is a fixed pt Let M be mid pt of PQ, then M is the pt where the common chord intersects PQ OM is always parallel to AP, and is drawn through the fixed pt O locus of M is a st line through O 7 Expressions are given in the text for the lengths of these tangents 8 Prove $\angle BCL=\angle BAC$, BC is a tangent to the circumcircle of AEC 9 The angles in each case are $\frac{1}{2}(B+C)$, $\frac{1}{2}(C+A)$, $\frac{1}{2}(A+B)$ 10 Let $2a$, $2b$ be the sides of the given rectangle Then radius of one of the inscribed circles$=\dfrac{ab}{a+b+\sqrt{a^2+b^2}}$ Sides of inner rectangle are found by subtracting twice this radius from each side of given rectangle. 11. $(AC+CB)^2=AC^2+CB^2+2AC \cdot CB$ 12 Draw the medians, notice that $2DG=AG$ Hence $2PD^2+PA^2=2DG^2+AG^2+3PG^2$ The locus is a circle, centre G 13 The centre of similitude of the fixed circles 14 The circle whose diameter is AB 15 Take D as origin, and let $DC=c$, $DB=b$, $DL=l$, where L is the foot of perp from A on BC, $AL=p$ Substitute these on both sides, noticing that $AD^2=p^2+l^2$, $AB^2=p^2+(b+l)^2$, $AC^2=p^2+(c-l)^2$ 16 Let D, E remain fixed while F moves to a very near pt F' Then for minimum area $\Delta DEF=\Delta DEF'$, FF' is parallel to DE, i e BC is parallel to DE 17 $\angle BCD=\angle DAE$=supplement of DFE 18 With the usual notation B and μ and $\frac{1}{2}R$ are given Since the N P circle is given, and line in which BC lies, therefore L and D are given With B as centre and rad $=R$ describe a circle, cutting the perp through D in G The circumcentre is known With O as centre and radius$=R$ describe a circle, cutting the base line in B, C and the altitude through L in A 19 By the general method for finding maxima and minima values, PQ must be along the line of centres in both cases 20 Let ABC be the Δ whose angles are given Suppose A is fixed, and B moves on the line BL. Draw $AL \perp BL$. On AL describe a ΔALP equiangular to ΔABC. Then ∵ AL is fixed,

P is a fixed pt Join CP Now $AL/AP=AB/AC$, $\therefore AL/AB=AP/AC$, and $\angle LAB=\angle PAC$, $\therefore$ Δs LAB, PAC are equiangular. Hence $\angle APC$ is a right angle But AP is fixed, PC is fixed 21 Particular case of last Ex 22 Let P be reqd pt. and Q a pt very near it on the line Then for a maximum we must have $\angle APB=\angle AQB$ A, B, P, Q are concyclic And $\because PQ$ is indefinitely small, $\because PQ$ is a tangent to the circle $APQB$ Hence describe a circle to pass through A, B and to touch L in the pt P [*Section IV*, *Prop V*] There are two solutions, hence two pts can be found at which $\angle APB$ is a maximum 23 Produce AB to meet L in P, then P is a position of zero minimum value 24 Follows from Ex 22 25 As in Ex 22 draw a circle to pass through A and B and to touch C in P [*Section IV*, *Prop IV*] Then P is the required pt 26 Δs APB, APC can be so placed as to form a cyclic quadrilateral 27 We have to prove that a centre of similitude lies on the bisector of the angle between the tangents at the pt of intersection Now the angle between tangents is the angle between the radii to the pt. of intersection And the bisector of the exterior angle between the radii divides the line of centres in the ratio of the radii 28 See Section I, Prop VI 29 The segments within the triangle are each capable of an angle of 120° 30 Let A be the given pt and L and M the given straight lines which are the loci of the angular pts B, C Since A is a fixed pt and B moves on L, the locus of C is a fixed line (Ex 20), let this locus cut the line M in C Then side AC is known 3 Apply Section I, Prop VI 32 Place anywhere in the circle a chord of the given length Describe a concentric circle touching this chord Through the given pt draw a tangent to this circle, terminated both ways by the outer circle 33 Construction similar to that of Ex. 32 34 Notice that I is the orthocentre of the triangle $I_1I_2I_3$ Prove that $\angle BIC=90°+\frac{A}{2}$, .. base and vertical angle of triangle BIC are given The locus of I is a circle Since $BICI_1$ is cyclic, locus of I_1 is the same circle Also prove that $\angle BI_2C=\frac{A}{2}=\angle BI_3C$ Hence the locus of I_2 and I_3 is the same circle 35 The result follows from similar triangles The formula is useful for calculating the perp when the sides are given 36 TP, TQ two tangents, OL, OM, ON perp to PQ, PT, QT. The quads. $OLQN$, $OLPM$ are cyclic Hence show that triangles OLM, OLN are equiangular 37 We have to prove that $BPQ=AQR$, *i.e.* $PQC=AQR$, *i e* $APC=APR$ 38 The incentre of the pedal

triangle is the orthocentre of the original 39 Let the tangent at P meet the given line in O Prove APB a right angle The locus is a circle on AB as diameter 40 Draw the two direct and one transverse common tangent of the circles 41 Draw two transverse tangents and one direct common tangent 42 The pt I is the orthocentre of the triangle $I_1I_2I_3$, and A, B, C are the feet of the altitudes of $I_1I_2I_3$ 43 Suppose the square inscribed as required, draw another square on the diameter, and notice that the centre of the semicircle is the homothetic centre of these two squares 44 Draw perps from the centre on the sides, each of these perps. is equal to the radius 45 and 46 $ABCDE$ hexagon, and P pt on circumference Let the perps from P on AB, BC, etc., be p_1, p_2, p_3 . p_6, and let PA, PB, PC be a, b, c, d, e, f Then if D be the diameter of the circle, we have $Dp_1=ab$, $Dp_2=bc$, $Dp_3=cd$, $Dp_4=de$, $Dp_5=ef$, $Dp_6=fa$ From these equations we get $p_1p_3p_5=p_2p_4p_6$ Again, if the alternate sides be made indefinitely small, they shall become the tangents at the angular pts. of an inscribed circle, and the theorem will still hold good. 47 As ACA_1, BCB_2 are congruent. Similarly other pairs. 48 Let $BD=x$, then $(a-x)$ $BD=x$ CD, where a=side of triangle, by Section III, Prop VIII, we have

$$(a-x)a^2+xa^2=(a-x)x^2+x(a-x)^2+a\ \ AD^2$$

$AD^2=ax+(a-x)^2$ Now $(a-x)^2$ is positive, $AD^2>ax>AB\ BD$ 49 Let ABC be the required Δ, whose sides CA, CB pass through the fixed pts P, Q, and whose centre is O O is the centre, $\angle PCO=\angle QCO=\angle PAO=\angle QBO=30°$ The locus of C is two segments on OP, OQ capable of $30°$ Hence C is determined The locus of A is a similar segment on OP Let CP produced cut this segment in A Thus A is determined Similarly, B can be determined 50 Proceed as in the proof of Bk. I, Prop. 27.

THE END